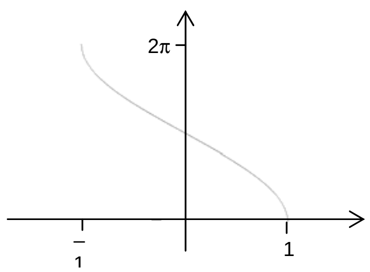
4753 (C3) Methods for Advanced Mathematics

Section A

1 $\int_0^{\pi/6} \sin 3x \, dx = \left[-\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$ $= -\frac{1}{3}\cos\frac{\pi}{2} + \frac{1}{3}\cos 0$ $= \frac{1}{3}$	B1 M1 A1cao [3]	$\left[-\frac{1}{3}\cos 3x\right]$ or $\left[-\frac{1}{3}\cos u\right]$ substituting correct limits in $\pm k$ **cos** … 0.33 or better.
2(i) $100 = Ae^0 = A \Rightarrow A = 100$ $50 = 100\,e^{-1500k}$ $\Rightarrow e^{-1500k} = 0.5$ $\Rightarrow -1500k = \ln 0.5$ $\Rightarrow k = -\ln 0.5 / 1500 = 4.62 \times 10^{-4}$	M1A1 M1 M1 A1 [5]	 $50 = A\,e^{-1500k}$ ft their 'A' if used taking lns correctly 0.00046 or better
(ii) $1 = 100e^{-kt}$ $\Rightarrow -kt = \ln 0.01$ $\Rightarrow t = -\ln 0.01 / k$ $= 9966$ years	M1 M1 A1 [3]	ft their A and k taking lns correctly art 9970
3 	M1 B1 A1 [3]	Can use degrees or radians reasonable shape (condone extra range) passes through (–1, 2π), (0, π) and (1, 0) good sketches – look for curve reasonably vertical at (–1, 2π) and (1, 0), negative gradient at (0, π). Domain and range must be clearly marked and correct.
4 $g(x) = 2\|x-1\|$ $\Rightarrow b = 2\|0-1\| = 2$ or (0, 2) $2\|x-1\| = 0$ $\Rightarrow x = 1$, so $a = 1$ or (1, 0)	 B1 M1 A1 [3]	Allow unsupported answers. www $\|x\| = 1$ is A0 www

5(i)	$e^{2y} = 1 + \sin x$ $\Rightarrow$ $2e^{2y}dy/dx = \cos x$ $\Rightarrow$ $dy/dx = \dfrac{\cos x}{2e^{2y}}$	M1 B1 A1 [3]	Their $2e^{2y} \times dy/dx$ $2e^{2y}$ o.e. cao
(ii)	$2y = \ln(1 + \sin x)$ $\Rightarrow$ $y = ½ \ln(1 + \sin x)$ $\Rightarrow$ $dy/dx = \dfrac{1}{2}\dfrac{\cos x}{1+\sin x}$ $= \dfrac{\cos x}{2e^{2y}}$ as before	 B1 M1 B1 E1 [4]	 chain rule (can be within 'correct' quotient rule with $dv/dx = 0$) $1/u$ or $1/(1 + \sin x)$ soi www
6	$\text{f f}(x) = \dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$ $= \dfrac{x+1+x-1}{x+1-x+1}$ $= 2x/2 = x$* $f^{-1}(x) = f(x)$ Symmetrical about $y = x$.	M1 M1 E1 B1 B1 [5]	correct expression without subsidiary denominators e.g. $= \dfrac{x+1+x-1}{x-1} \times \dfrac{x-1}{x+1-x+1}$ stated, or shown by inverting
7(i)	(A) $(x - y)(x^2 + xy + y^2)$ $= x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3$ $= x^3 - y^3$ * (B) $(x + ½ y)^2 + ¾ y^2$ $= x^2 + xy + ¼ y^2 + ¾ y^2$ $= x^2 + xy + y^2$	M1 E1 M1 E1 [4]	expanding - allow tabulation www $(x + ½ y)^2 = x^2 + ½ xy + ½ xy + ¼ y^2$ o.e. cao www
(ii)	$x^3 - y^3 = (x - y)[(x + ½ y)^2 + ¾ y^2]$ $(x + ½ y)^2 + ¾ y^2 > 0$ [as squares ≥ 0] $\Rightarrow$ if $x - y > 0$ then $x^3 - y^3 > 0$ $\Rightarrow$ if $x > y$ then $x^3 > y^3$ *	M1 M1 E1 [3]	substituting results of (i)

8(i) A: $1 + \ln x = 0$ $\Rightarrow$ $\ln x = -1$ so A is $(e^{-1}, 0)$ $\Rightarrow$ $x = e^{-1}$ B: $x = 0, y = e^{0-1} = e^{-1}$ so B is $(0, e^{-1})$ C: $f(1) = e^{1-1} = e^0 = 1$ $g(1) = 1 + \ln 1 = 1$	M1 A1 B1 E1 E1 [5]	 SC1 if obtained using symmetry condone use of symmetry Penalise A = e^{-1}, B = e^{-1}, or co-ords wrong way round, but condone labelling errors.
(ii) *Either* by invertion: e.g. $y = e^{x-1}$ $x \leftrightarrow y$ $x = e^{y-1}$ $\Rightarrow$ $\ln x = y - 1$ $\Rightarrow$ $1 + \ln x = y$	 M1 E1	 taking lns or exps
or by composing e.g. $f\,g(x) = f(1 + \ln x)$ $= e^{1 + \ln x - 1}$ $= e^{\ln x} = x$	 M1 E1 [2]	 $e^{1 + \ln x - 1}$ or $1 + \ln(e^{x-1})$
(iii) $\int_0^1 e^{x-1}dx = \left[e^{x-1}\right]_0^1$ $= e^0 - e^{-1}$ $= 1 - e^{-1}$	M1 M1 A1cao [3]	$\left[e^{x-1}\right]$ o.e or $u = x - 1 \Rightarrow \left[e^u\right]$ substituting correct limits for x or u o.e. not e^0, must be exact.
(iv) $\int \ln x dx = \int \ln x \frac{d}{dx}(x)dx$ $= x\ln x - \int x.\frac{1}{x}dx$ $= x \ln x - x + c$ $\Rightarrow$ $\int_{e^{-1}}^1 g(x)dx = \int_{e^{-1}}^1 (1+\ln x)dx$ $= \left[x + x\ln x - x\right]_{e^{-1}}^1$ $= \left[x\ln x)\right]_{e^{-1}}^1$ $= 1\ln 1 - e^{-1}\ln(e^{-1})$ $= e^{-1}$ *	M1 A1 A1cao B1ft DM1 E1 [6]	parts: $u = \ln x$, $du/dx = 1/x$, $v = x$, $dv/dx = 1$ condone no 'c' ft their '$x \ln x - x$' (provided 'algebraic') substituting limits dep B1 www
(v) Area = $\int_0^1 f(x)\,d\,x - \int_{e^{-1}}^1 g(x)\,d\,x$ $= (1 - e^{-1}) - e^{-1}$ $= 1 - 2/e$	M1 A1cao	Must have correct limits 0.264 or better.
or Area OCB = area under curve – triangle $= 1 - e^{-1} - ½ \times 1 \times 1$ $= ½ - e^{-1}$ *or* Area OAC = triangle – area under curve $= ½ \times 1 \times 1 - e^{-1}$ $= ½ - e^{-1}$ Total area = $2(½ - e^{-1}) = 1 - 2/e$	 M1 A1cao [2]	 OCA or OCB = $½ - e^{-1}$ 0.264 or better

9(i) $a = 1/3$	B1 [1]	or 0.33 or better
(ii) $\frac{dy}{dx} = \frac{(3x-1)2x - x^2.3}{(3x-1)^2}$ $= \frac{6x^2 - 2x - 3x^2}{(3x-1)^2}$ $= \frac{3x^2 - 2x}{(3x-1)^2}$ $= \frac{x(3x-2)}{(3x-1)^2}$ *	M1 A1 E1 [3]	quotient rule www – must show both steps; penalise missing brackets.
(iii) $dy/dx = 0$ when $x(3x-2) = 0$ $\Rightarrow$ $x = 0$ or $x = 2/3$, so at P, $x = 2/3$ when $x = \frac{2}{3}$, $y = \frac{(2/3)^2}{3\times(2/3)-1} = \frac{4}{9}$ when $x = 0.6$, $dy/dx = -0.1875$ when $x = 0.8$, $dy/dx = 0.1633$ Gradient increasing $\Rightarrow$ minimum	M1 A1 M1 A1cao B1 B1 E1 [7]	if denom = 0 also then M0 o.e e.g. $0.\dot{6}$, but must be exact o.e e.g. $0.\dot{4}$, but must be exact $-3/16$, or -0.19 or better 8/49 or 0.16 or better o.e. e.g. 'from negative to positive'. Allow ft on their gradients, provided –ve and +ve respectively. Accept table with indications of signs of gradient.
(iv) $\int \frac{x^2}{3x-1}dx$ $u = 3x-1 \Rightarrow du = 3dx$ $= \int \frac{\frac{(u+1)^2}{9}}{u}\frac{1}{3}du$ $= \frac{1}{27}\int \frac{(u+1)^2}{u}du = \frac{1}{27}\int \frac{u^2+2u+1}{u}du$ $= \frac{1}{27}\int (u+2+\frac{1}{u})du$ * Area = $\int_{2/3}^{1} \frac{x^2}{3x-1}dx$ When $x = 2/3$, $u = 1$, when $x = 1$, $u = 2$ $= \frac{1}{27}\int_1^2 (u+2+1/u)du$ $= \frac{1}{27}\left[\frac{1}{2}u^2 + 2u + \ln u\right]_1^2$ $= \frac{1}{27}[(2+4+\ln 2) - (\frac{1}{2}+2+\ln 1)]$ $= \frac{1}{27}(3\frac{1}{2}+\ln 2)$ $[= \frac{7+2\ln 2}{54}]$	B1 M1 M1 E1 B1 M1 A1cao [7]	$\frac{\frac{(u+1)^2}{9}}{u}$ o.e. × 1/3 (du) expanding Condone missing du's $\left[\frac{1}{2}u^2 + 2u + \ln u\right]$ substituting correct limits, dep integration o.e., but must evaluate ln 1 = 0 and collect terms.