OCR
Oxford Cambridge and RSA

Monday 19 October 2020 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \dfrac{u}{v}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$ where $q = 1 - p$

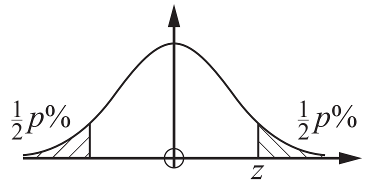
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 Find the value of $\sum_{r=1}^{5} 2^r(r-1)$. **[2]**

2 The graph of $y = |1-x| - 2$ is shown in Fig. 2.

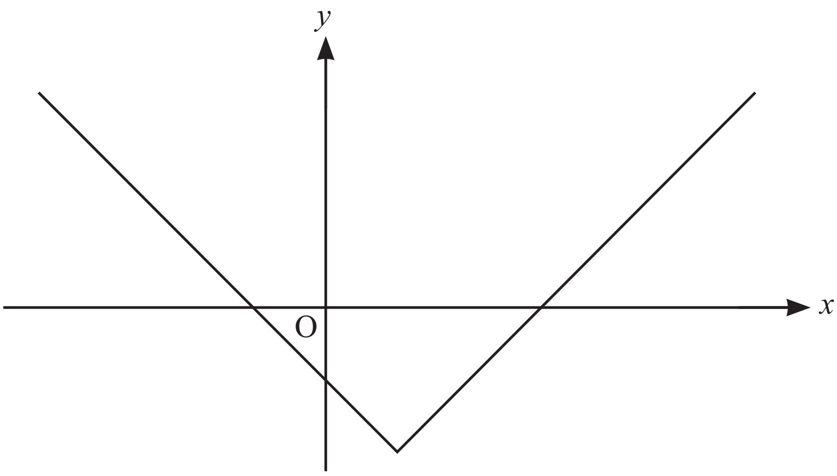


Fig. 2

Determine the set of values of x for which $|1-x| > 2$. **[4]**

3 A particular phone battery will last 10 hours when it is first used. Every time it is recharged, it will only last 98% of its previous time.

Find the maximum total length of use for the battery. **[3]**

4 Fig. 4 shows the regular octagon ABCDEFGH.

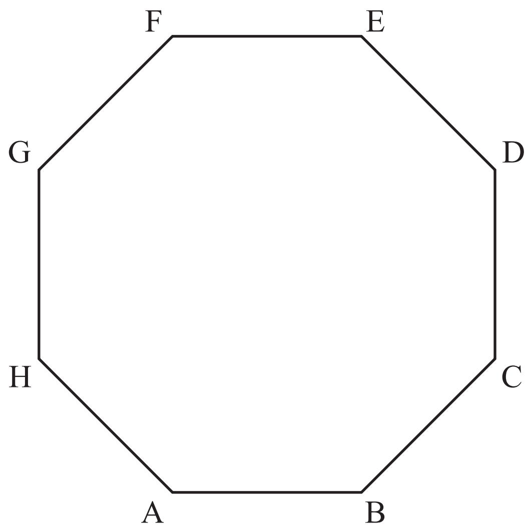


Fig. 4

$\overrightarrow{AB} = \mathbf{i}$, $\overrightarrow{CD} = \mathbf{j}$, where $\mathbf{i}$ is a unit vector parallel to the x-axis and $\mathbf{j}$ is a unit vector parallel to the y-axis.

Find an exact expression for $\overrightarrow{BC}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. **[3]**

5 Fig. 5 shows part of the curve $y = \operatorname{cosec} x$ together with the x- and y-axes.

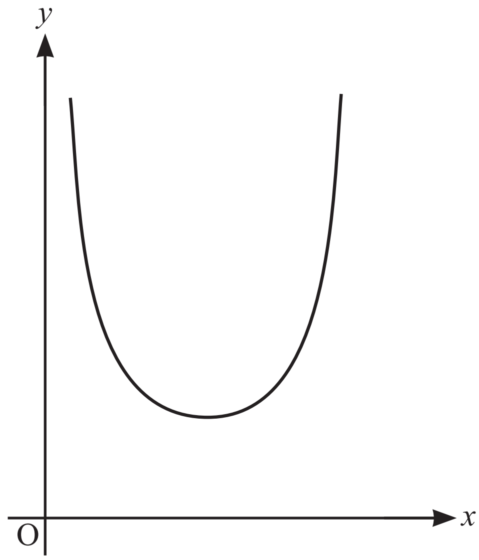


Fig. 5

(a) For the section of the curve which is shown in Fig. 5, write down

(i) the equations of the two vertical asymptotes, **[2]**

(ii) the coordinates of the minimum point. **[1]**

(b) Show that the equation $x = \operatorname{cosec} x$ has a root which lies between $x = 1$ and $x = 2$. **[2]**

(c) Use the iteration $x_{n+1} = \operatorname{cosec}(x_n)$, with $x_0 = 1$, to find

(i) the values of x_1 and x_2, correct to 5 decimal places, **[1]**

(ii) this root of the equation, correct to 3 decimal places. **[1]**

(d) There is another root of $x = \operatorname{cosec} x$ which lies between $x = 2$ and $x = 3$.

Determine whether the iteration $x_{n+1} = \operatorname{cosec}(x_n)$ with $x_0 = 2.5$ converges to this root. **[1]**

(e) Sketch the staircase or cobweb diagram for the iteration, starting with $x_0 = 2.5$, on the diagram in the Printed Answer Booklet. **[3]**

6 **(a)** **(i)** Write down the derivative of e^{kx}, where k is a constant. **[1]**

(ii) A business has been running since 2009. They sell maths revision resources online.

Give a reason why an exponential growth model might be suitable for the annual profits for the business. **[1]**

Fig. 6 shows the relationship between the annual profits of the business in thousands of pounds (y) and the time in years after 2009 (x). The graph of $\ln y$ plotted against x is approximately a straight line.

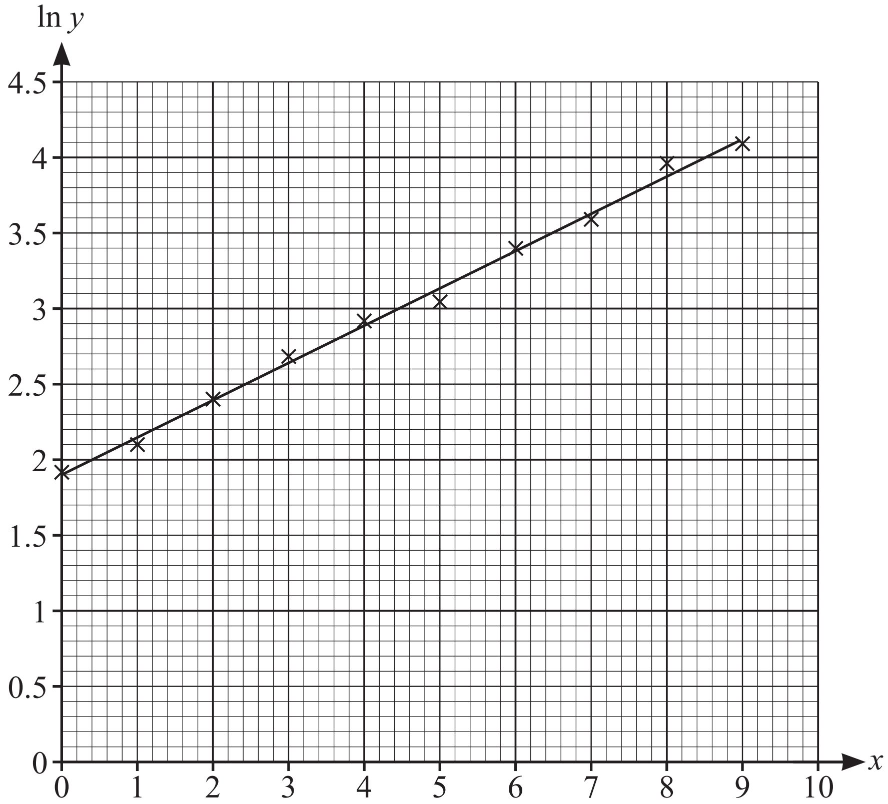


Fig. 6

(b) Show that the straight line is consistent with a model of the form $y = Ae^{kx}$, where A and k are constants. **[2]**

(c) Estimate the values of A and k. **[4]**

(d) Use the model to predict the profit in the year 2020. **[3]**

(e) How reliable do you expect the prediction in part **(d)** to be? Justify your answer. **[1]**

7 **(a)** Express $\frac{1}{x}+\frac{1}{A-x}$ as a single fraction. **[1]**

The population of fish in a lake is modelled by the differential equation

$$\frac{dx}{dt}=\frac{x(400-x)}{400}$$

where x is the number of fish and t is the time in years.

When $t=0$, $x=100$.

(b) **In this question you must show detailed reasoning.**

Find the number of fish in the lake when $t=10$, as predicted by the model. **[8]**

8 **(a)** The curve $y=\frac{1}{(1+x^2)^2}$ is shown in Fig. 8.

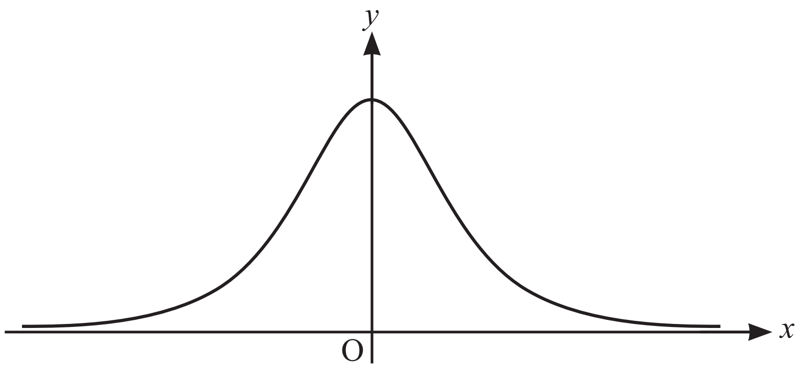


Fig. 8

(i) Show that $\frac{d^2y}{dx^2}=\frac{20x^2-4}{(1+x^2)^4}$. **[5]**

(ii) **In this question you must show detailed reasoning.**

Find the set of values of x for which the curve is concave downwards. **[3]**

(b) Use the substitution $x=\tan\theta$ to find the exact value of $\int_{-1}^{1}\frac{1}{(1+x^2)^2}\,dx$. **[8]**

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

9 **(a)** Show that if $a = 1$ and $b > 1$ then $a^b < b^a$. **[2]**

(b) Find integer values of a and b with $b > a > 1$ and a^b not greater than b^a (a counter example to the conjecture given in lines 7–8). **[1]**

10 **In this question you must show detailed reasoning.**

Show that $\int_{\mathrm{e}}^{\pi} \frac{1}{x}\,\mathrm{d}x = \ln\pi - 1$ as given in line 37. **[2]**

11 Show that e^x is an increasing function for all values of x, as stated in line 39. **[2]**

12 **(a)** Show that the only stationary point on the curve $y = \frac{\ln x}{x}$ occurs where $x = \mathrm{e}$, as given in line 45. **[3]**

(b) Show that the stationary point is a maximum. **[3]**

(c) It follows from part **(b)** that, for any positive number a with $a \neq \mathrm{e}$,

$$\frac{\ln \mathrm{e}}{\mathrm{e}} > \frac{\ln a}{a}.$$

Use this fact to show that $\mathrm{e}^a > a^{\mathrm{e}}$. **[2]**

END OF QUESTION PAPER

BLANK PAGE

BLANK PAGE

Oxford Cambridge and RSA

Monday 19 October 2020 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension Insert

Time allowed: 2 hours

8168137993

INSTRUCTIONS
- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION
- This Insert contains the article for Section B.
- This document has **4** pages.

Which is bigger?

Which is bigger: π^{e} or e^{π}? Using a calculator confirms that e^{π} is the larger, but how can this be proved without the use of a calculator?

Simpler examples

It is often helpful in mathematics to consider simpler examples. It is easy to work out that $3^4 > 4^3$. In the expression 3^4, 3 is the base and 4 is the exponent. Working with integers greater than 1, it is easy to find many examples where $a^b > b^a$ if $a < b$. That is, using the smaller base and the larger exponent gives the larger result. This might lead us to conjecture that $a^b > b^a$ if $a < b$ and both a and b are integers greater than 1. However, it is also possible to find counter examples to this conjecture.

Exponents can also be rational numbers, and in general $x^{\frac{p}{q}}$ denotes $\left(\sqrt[q]{x}\right)^p$ where p and q are integers and q is positive. So, any rational power of a positive number, x, can be defined. However, both e and π are irrational numbers. Considering the original question about π^{e} and e^{π} raises the issue of what is meant by an irrational power of a number.

Extending the definition of power to irrational numbers

What, for example, is meant by 2^{π}?

An irrational number corresponds to a non-recurring infinite decimal. Rounding the decimal gives a rational approximation to the irrational number. For example, the following sequence gives increasingly accurate approximations to π.

3, 3.1, 3.14, 3.142, 3.1416, 3.14159, …

Using a spreadsheet gives a sequence of approximations to 2^{π}, as shown in Fig. C1. The limit of this sequence of approximations is the value of 2^{π}. This limit cannot be evaluated with a spreadsheet but it is, in principle, possible to find the value to any required degree of accuracy.

	A	B
1	k	2^k
2	3	8
3	3.1	8.574188
4	3.14	8.815241
5	3.142	8.82747
6	3.1416	8.825023
7	3.14159	8.824962

Fig. C1

2^x and x^2 are increasing functions of x for $x > 0$ and this allows us to deduce that $\pi^2 > 2^{\pi}$, as follows.

We know that π is between 3 and 3.142
$\pi < 3.142 \Rightarrow 2^{\pi} < 2^{3.142} = 8.82747$
$\pi > 3 \Rightarrow \pi^2 > 3^2 = 9$
So $\pi^2 > 9 > 8.82747 > 2^{\pi}$
Hence $\pi^2 > 2^{\pi}$

Which is bigger: π^{e} or e^{π}?

An indirect method, using calculus, enables us to prove that e^{π} is larger than π^{e}. Fig. C2 shows the curve $y = \frac{1}{x}$ in the first quadrant together with the rectangle with vertices at the points (e, 0), $\left(e, \frac{1}{e}\right)$, $\left(\pi, \frac{1}{e}\right)$ and $(\pi, 0)$. We use the fact that the area under the curve between e and π is less than the area of this rectangle.

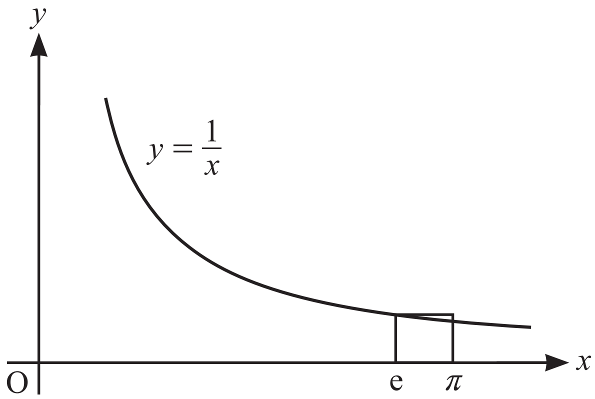


Fig. C2

The area of the rectangle is $\frac{1}{e}(\pi - e)$

$\int_{e}^{\pi} \frac{1}{x}\,dx < \frac{1}{e}(\pi - e)$

$\ln \pi - 1 < \frac{\pi}{e} - 1$

$\ln \pi < \frac{\pi}{e}$

e^{x} is an increasing function for all values of x

hence $\pi < e^{\frac{\pi}{e}}$

Assuming that the usual rules of indices apply to irrational powers of irrational numbers, raising both sides of the inequality to the power e gives the desired result.

Using a similar method, it can be shown that $e^{a} > a^{e}$ for any positive number $a \neq e$.

An alternative method for showing that $e^{a} > a^{e}$ for any positive number a is to show that the only stationary point on the curve $y = \frac{\ln x}{x}$ (a maximum) occurs where $x = e$.