Oxford Cambridge and RSA

Monday 4 October 2021 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

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INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Section A (31 marks)

Answer **all** the questions.

1 **(a)** Express $\frac{1}{(2r-1)(2r+1)}$ in partial fractions. **[3]**

(b) Hence find $\sum_{r=1}^{n}\frac{1}{(2r-1)(2r+1)}$, expressing the result as a single fraction. **[4]**

2 **In this question you must show detailed reasoning.**

Find the gradient of the curve $y = 6\arcsin(2x)$ at the point with x-coordinate $\frac{1}{4}$. Express the result in the form $m\sqrt{n}$, where m and n are integers. **[4]**

3 **In this question you must show detailed reasoning.**

The complex numbers z_1 and z_2 are given by $z_1 = -2+2\text{i}$ and $z_2 = 2\left(\cos\frac{1}{6}\pi + \text{i}\sin\frac{1}{6}\pi\right)$.

(a) Find the modulus and argument of z_1. **[2]**

(b) Hence express $\frac{z_1}{z_2}$ in exact modulus-argument form. **[4]**

4 **In this question you must show detailed reasoning.**

Determine the mean value of $\frac{1}{1+4x^2}$ between $x = -1$ and $x = 1$. Give your answer to **3** significant figures. **[4]**

5 **(a)** Use a Maclaurin series to find a quadratic approximation for $l\ \ (1+2x)$. **[1]**

(b) Find the percentage error in using the approximation in part **(a)** to calculate $\ln(1.2)$. **[3]**

(c) Jane uses the Maclaurin series in part **(a)** to try to calculate an approximation for $\ln 3$.

Explain whether her method is valid. **[2]**

6 Given that $y = mx$ is an invariant line of the transformation with matrix $\begin{pmatrix}1 & 2\\ 2 & -2\end{pmatrix}$, determine the possible values of m. **[4]**

Section B (113 marks)

Answer **all** the questions.

7 Prove that $\sum_{r=1}^{n} \frac{r}{2^{r-1}} = 4 - \frac{n+2}{2^{n-1}}$ for all $n \geqslant 1$. **[6]**

8 The equation $4x^4 - 4x^3 + px^2 + qx - 9 = 0$, where p and q are constants, has roots α, $-\alpha$, β and $\frac{1}{\beta}$.

(a) Determine the exact roots of the equation. **[5]**

(b) Determine the values of p and q. **[4]**

9 The transformation T of the plane has associated matrix **M**, where $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$.

(a) On the grid in the Printed Answer Booklet, plot the image OA′B′C′ of the unit square OABC under the transformation T. **[2]**

(b) **(i)** Calculate the value of det **M**. **[1]**

(ii) Explain the significance of the value of det **M** in relation to the image OA′B′C′. **[2]**

(c) T is equivalent to a sequence of two transformations of the plane.

(i) Specify fully **two** transformations equivalent to T. **[3]**

(ii) Use matrices to verify your answer. **[3]**

10 **(a)** Show on an Argand diagram the points representing the three cube roots of unity. **[2]**

(b) **(i)** Find the exact roots of the equation $z^3 - 1 = \sqrt{3}\,\mathrm{i}$, expressing them in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta < \pi$. **[5]**

(ii) The points representing the cube roots of unity form a triangle Δ_1. The points representing the roots of the equation $z^3 - 1 = \sqrt{3}\,\mathrm{i}$ form a triangle Δ_2.

State a sequence of two transformations that maps Δ_1 onto Δ_2. **[2]**

(iii) The three roots in part **(b)(i)** are z_1, z_2 and z_3.

By simplifying $z_1 + z_2 + z_3$, verify that the sum of these roots is zero. **[2]**

(iv) Hence show that $\sin 20° + \sin 140° = \sin 100°$. **[2]**

11 **(a)** Given that $\mathbf{u} = \lambda\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, find the following, giving your answers in terms of λ.

(i) $\mathbf{u}.\mathbf{v}$ **[1]**

(ii) $\mathbf{u} \times \mathbf{v}$ **[2]**

(b) Hence determine

(i) the acute angle between the planes $2x + y - 3z = 10$ and $x + 2y - 2z = 10$, **[3]**

(ii) the shortest distance between the lines $\frac{x-3}{3} = \frac{y}{1} = \frac{z-2}{-3}$ and $\frac{x}{1} = \frac{y-4}{2} = \frac{z+2}{-2}$, giving your answer as a multiple of $\sqrt{2}$. **[3]**

12 **Fig. 12** shows a rhombus OACB in an Argand diagram. The points A and B represent the complex numbers z and w respectively.

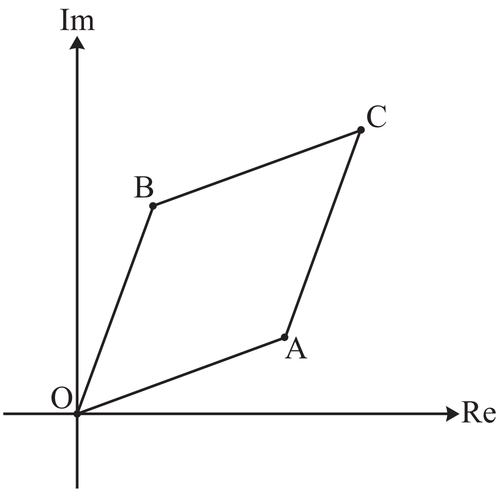

Fig. 12

Prove that $\arg(z + w) = \frac{1}{2}(\arg z + \arg w)$.

[A copy of **Fig. 12** is provided in the Printed Answer Booklet.] **[4]**

13 Find the general solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 2\mathrm{e}^x$. **[7]**

14 A curve has polar equation $r = a(\cos\theta + 2\sin\theta)$, where a is a positive constant and $0 \leqslant \theta \leqslant \pi$.

(a) Determine the polar coordinates of the point on the curve which is furthest from the pole. **[7]**

(b) **(i)** Show that the curve is a circle whose radius should be specified. **[6]**

(ii) Write down the polar coordinates of the centre of the circle. **[1]**

15 The equations of three planes are

$$\begin{aligned} -4x + ky + 7z &= 4, \\ x - 2y + 5z &= l, \\ 2x + 3y + z &= 2. \end{aligned}$$

Given that the planes form a sheaf, determine the values of k and l. **[6]**

16 **(a)** Show using exponentials that $\cosh 2u = 1 + 2\sinh^2 u$. **[4]**

(b) Show that $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4+x^2}}\,\mathrm{d}x = 2\sqrt{2} - 2\ln(1+\sqrt{2})$. **[10]**

17 In a chemical process, a vessel contains 1 litre of pure water. A liquid chemical is then passed into the top of the vessel at a constant rate of a litres per minute and thoroughly mixed with the water. At the same time, the resulting mixture is drawn from the bottom of the vessel at a constant rate of b litres per minute. You may assume that the chemical mixes instantly and uniformly with the water. After t minutes, the mixture in the vessel contains x litres of the chemical.

(a) **(i)** Show that the proportion of chemical present in the vessel after t minutes is $\dfrac{x}{1+(a-b)t}$. **[2]**

(ii) Hence show that $\dfrac{\mathrm{d}x}{\mathrm{d}t} + \dfrac{bx}{1+(a-b)t} = a$. **[2]**

(b) First, consider the case where $b = a$.

(i) Solve the differential equation to find x in terms of a and t. **[4]**

(ii) Given that after 1 minute the vessel contains equal amounts of water and chemical, find the rate of inflow of chemical. **[2]**

(c) Now consider the case where $b = 2a$.

(i) Explain why the differential equation in part **(a)(ii)** is now invalid for $t \geqslant \dfrac{1}{a}$. **[1]**

(ii) Find the maximum amount of chemical in the vessel. **[9]**

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