OCR
Oxford Cambridge and RSA

Tuesday 20 June 2023 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

9918376966

You must have:
- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$$S_n = \tfrac{1}{2}n(a+l) = \tfrac{1}{2}n\{2a+(n-1)d\}$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Binomial series

$$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \dfrac{u}{v}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x} - u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

Integration

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

$$\int \mathrm{f}'(x)\big(\mathrm{f}(x)\big)^n\,\mathrm{d}x = \frac{1}{n+1}\big(\mathrm{f}(x)\big)^{n+1} + c$$

Integration by parts $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \tfrac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \mathrm{si}\quad A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r p^r q^{n-r}$ where $q = 1 - p$

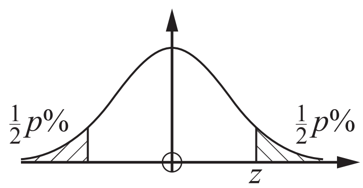
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

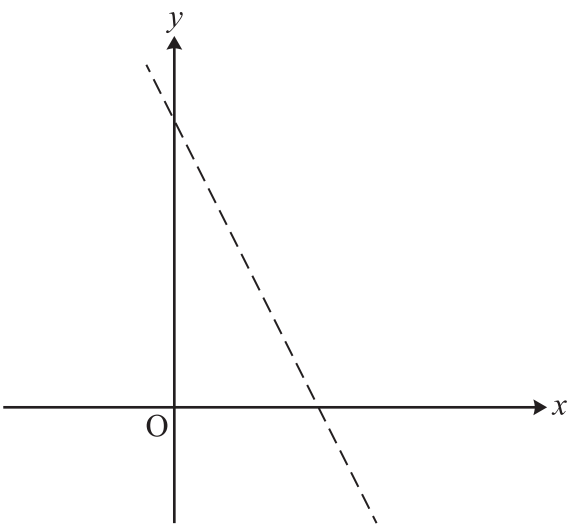
Section A (60 marks)

1 **In this question you must show detailed reasoning.**

The obtuse angle θ is such that $\sin\theta = \dfrac{2}{\sqrt{13}}$.

Find the exact value of $\cos\theta$. **[3]**

2 The straight line $y = 5 - 2x$ is shown in the diagram.



(a) On the copy of the diagram in the Printed Answer Booklet, sketch the graph of $y = |5 - 2x|$. **[1]**

(b) Solve the inequality $|5 - 2x| < 3$. **[3]**

3 **In this question you must show detailed reasoning.**

Find the value of k such that $\dfrac{1}{\sqrt{5}+\sqrt{6}} + \dfrac{1}{\sqrt{6}+\sqrt{7}} = \dfrac{k}{\sqrt{5}+\sqrt{7}}$. **[3]**

4 In this question you must show detailed reasoning.

Find the coordinates of the points where the curve $y = x^3 - 2x^2 - 5x + 6$ crosses the x-axis. **[4]**

5 In this question you must show detailed reasoning.

This question is about the curve $y = x^3 - 5x^2 + 6x$.

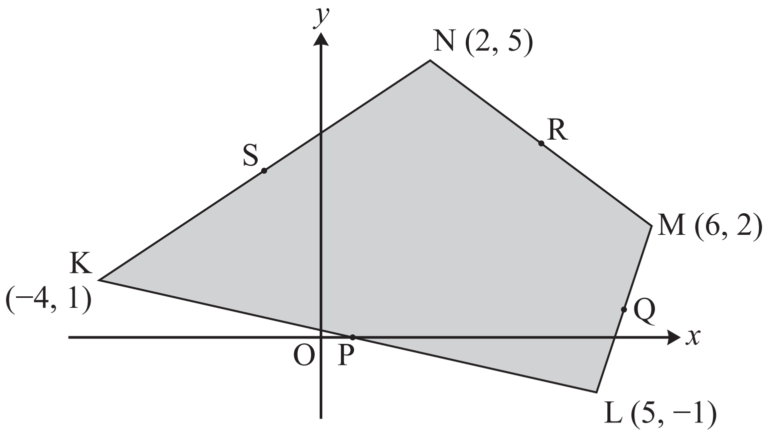
(a) Find the equation of the tangent, T, to the curve at the point (0, 0). **[3]**

(b) Find the equation of the normal, N, to the curve at the point (1, 2). **[3]**

(c) Find the coordinates of the point of intersection of T and N. **[2]**

6 **(a)** Quadrilateral KLMN has vertices K $(-4, 1)$, L $(5, -1)$, M $(6, 2)$ and N $(2, 5)$, as shown in **Fig. 6.1**.

Fig. 6.1



(i) Find the coordinates of the following midpoints.

- P, the midpoint of KL
- Q, the midpoint of LM
- R, the midpoint of MN
- S, the midpoint of NK **[2]**

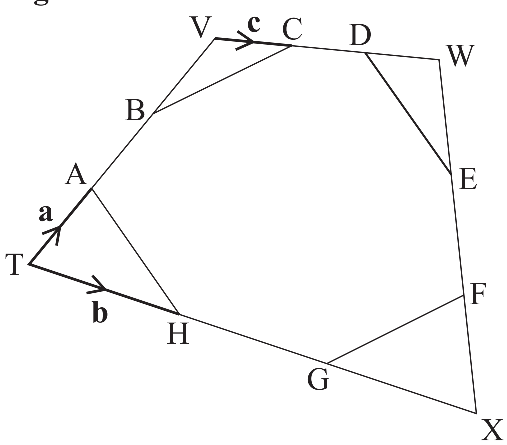
(ii) Verify that PQRS is a parallelogram. **[3]**

(b) TVWX is a quadrilateral as shown in **Fig. 6.2**.

Points A and B divide side TV into 3 equal parts. Points C and D divide side VW into 3 equal parts. Points E and F divide side WX into 3 equal parts. Points G and H divide side TX into 3 equal parts.

$\overrightarrow{TA} = \mathbf{a}$, $\overrightarrow{TH} = \mathbf{b}$, $\overrightarrow{VC} = \mathbf{c}$.

Fig. 6.2

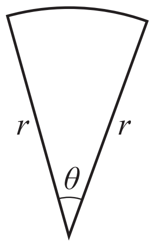


(i) Show that $\overrightarrow{WX} = k(-\mathbf{a} + \mathbf{b} - \mathbf{c})$, where k is a constant to be determined. **[1]**

(ii) Verify that AH is parallel to DE. **[2]**

(iii) Verify that BC is parallel to GF. **[2]**

7 A wire, 10 cm long, is bent to form the perimeter of a sector of a circle, as shown in the diagram. The radius is r cm and the angle at the centre is θ radians.



Determine the maximum possible area of the sector, showing that it is a maximum. **[6]**

8 A circle with centre A and radius 8 cm and a circle with centre C and radius 12 cm intersect at points B and D.

Quadrilateral ABCD has area $60\,\text{cm}^2$.

Determine the two possible values for the length AC. **[7]**

9 A small country started using solar panels to produce electrical energy in the year 2000. Electricity production is measured in megawatt hours (MWh).

For the period from 2000 to 2009, the annual electrical energy produced using solar panels can be modelled by the equation $P = 0.3e^{0.5t}$, where P is the annual amount of electricity produced in MWh and t is the time in years after the year 2000.

(a) According to this model, find the amount of electricity produced using solar panels in each of the following years.

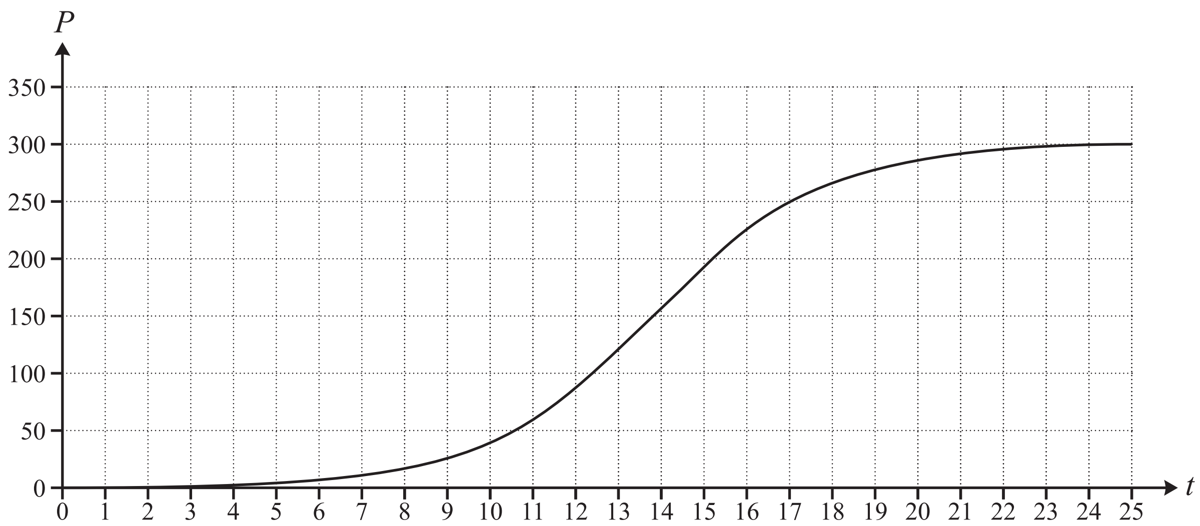
(i) 2000 **[1]**

(ii) 2009 **[1]**

(b) Give a reason why the model is unlikely to be suitable for predicting the annual amount of electricity produced using solar panels in the year 2025. **[1]**

An alternative model is suggested; the curve representing this model is shown in **Fig. 9**.

Fig. 9



(c) Explain how the graph shows that the alternative model gives a value for the amount of electricity produced in 2009 that is consistent with the original model. **[1]**

(d) **(i)** On the axes given in the Printed Answer Booklet, sketch the gradient function of the model shown in **Fig. 9**. **[2]**

(ii) State approximately the value of t at the point of inflection in **Fig. 9**. **[1]**

(iii) Interpret the significance of the point of inflection in the context of the model. **[1]**

(e) State approximately the long term value of the annual amount of electricity produced using solar panels according to the model represented in **Fig. 9**. **[1]**

10 (a) You are given that $(x^2+y^2)^3 = x^6+3x^4y^2+3x^2y^4+y^6$.

Hence, or otherwise, prove that $\sin^6\theta+\cos^6\theta = 1-\frac{3}{4}\sin^2 2\theta$ for all values of θ. **[4]**

(b) Use the result from part **(a)** to determine the minimum value of $\sin^6\theta+\cos^6\theta$. **[2]**

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

11 **(a)** Evaluate $\sum_{r=1}^{5} r^2$. **[1]**

(b) Show that Euler's approximate formula, as given in line 13, gives the exact value of $\sum_{r=1}^{5} r^2$. **[2]**

12 With the aid of a suitable diagram, show that the three triangles referred to in line 26 have the areas given in line 27. **[3]**

13 Prove that Euler's approximate formula, as given in line 13, when applied to $\sum_{r=1}^{n} r^2$ gives exactly $\dfrac{n(n+1)(2n+1)}{6}$. **[4]**

14 Show that the expression given in line 33 simplifies to $\displaystyle\sum_{r=1}^{n} \frac{1}{r} \approx 1\ \ n + \frac{13}{24} + \frac{6n+5}{12n(n+1)}$, as given in line 34. **[3]**

15 The expression given in line 34 is used to calculate $\displaystyle\sum_{r=1}^{6} \frac{1}{r}$.

Show that the error in the result is less than 1.5% of the true value. **[2]**

END OF QUESTION PAPER

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Insert

Time allowed: 2 hours

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INSTRUCTIONS

- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

- This Insert contains the article for Section B.
- This document has **4** pages.

Approximating series

Powers of natural numbers

The sum of the first n natural numbers, $1+2+3+\ldots+n$, can be worked out using the formula for the sum of an arithmetic series.

The sum of the squares of the first n natural numbers, $1^2+2^2+3^2+\ldots+n^2$, can be expressed exactly as a formula, $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. There are also exact formulae for the sum of the cubes and for higher powers.

However, the sum of the reciprocals of the first n natural numbers, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, cannot be expressed exactly in terms of n and only approximate formulae can be found. This particular series is called the harmonic series.

Euler’s approximate summation formula

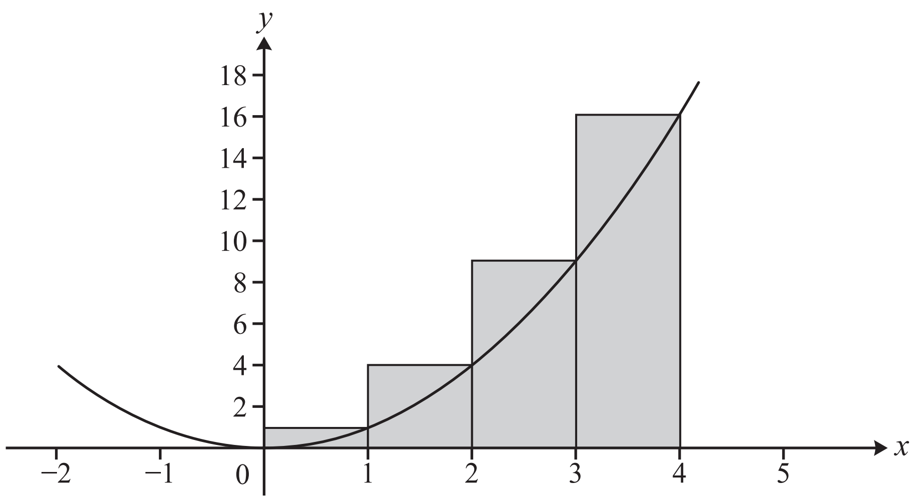
In 1741, the mathematician Leonhard Euler published an approximate formula for summing a series. In modern notation, this can be expressed as follows.

$$\sum_{r=1}^{n} \mathrm{f}(r) \approx \int_1^n \mathrm{f}(x)\,\mathrm{d}x + \frac{\mathrm{f}(n)+\mathrm{f}(1)}{2} + \frac{\mathrm{f}(1)-\mathrm{f}(2)}{12} - \frac{\mathrm{f}(n)-\mathrm{f}(n+1)}{12}$$

Exploring Euler’s approximate summation formula for sums of squares

Using Euler’s approximate formula for the sum of squares of natural numbers gives the exact sum of the series.

Euler’s formula relates a sum of terms to an integral, and this can be illustrated by considering a suitable graph. For the sum of the squares of natural numbers, this is the graph of $y = x^2$. The diagram shows this curve, with four shaded rectangles of areas 1^2, 2^2, 3^2 and 4^2.



Euler's approximate formula for this case is as follows.

$$\sum_{r=1}^{4} r^2 \approx \int_1^4 x^2\,\mathrm{d}x + \frac{4^2+1^2}{2} + \frac{1^2-2^2}{12} - \frac{4^2-5^2}{12}$$

The integral gives the area under the curve between $x = 1$ and $x = 4$. It is clear that the integral is smaller than $\sum_{r=1}^{4} r^2$ so something needs to be added to the integral to get the same answer as the series. The rectangle for 1^2 needs to be added on and so do the parts of the other three rectangles that are above the curve.

Approximating the curve by a series of straight lines gives three triangles to be added on. These have areas $\frac{2^2-1^2}{2}$, $\frac{3^2-2^2}{2}$ and $\frac{4^2-3^2}{2}$.

This gives an approximation for the series of $\int_1^4 x^2\,\mathrm{d}x + 1^2 + \frac{2^2-1^2}{2} + \frac{3^2-2^2}{2} + \frac{4^2-3^2}{2}$ which simplifies to $\int_1^4 x^2\,\mathrm{d}x + \frac{4^2+1^2}{2}$. The final two terms in Euler's approximate formula are to correct for the curve not being a series of straight lines.

Applying Euler's approximate summation formula to the harmonic series

Using Euler's approximate summation for the harmonic series gives

$$\sum_{r=1}^{n} \frac{1}{r} \approx \int_1^n \frac{1}{x}\,\mathrm{d}x + \frac{1}{2}\left(\frac{1}{n}+1\right) + \frac{1}{12}\left(1-\frac{1}{2}\right) - \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

This simplifies to $\sum_{r=1}^{n} \frac{1}{r} \approx 1\;\; n + \frac{13}{24} + \frac{6n+5}{12n(n+1)}$.