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**ADVANCED GCE**

# MATHEMATICS (MEI) 4754A

Applications of Advanced Mathematics (C4) Paper A

Candidates answer on the answer booklet.

**OCR supplied materials:**

- 8 page answer booklet (sent with general stationery)
- MEI Examination Formulae and Tables (MF2)

**Other materials required:**

- Scientific or graphical calculator

**Friday 14 January 2011**
**Afternoon**

**Duration:** 1 hour 30 minutes

## INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

## INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

## NOTE

- This paper will be followed by **Paper B: Comprehension**.

**Section A** (36 marks)

**1** **(i)** Use the trapezium rule with four strips to estimate $\int_{-2}^{2} \sqrt{1+e^x}\, dx$, showing your working. **[4]**

Fig. 1 shows a sketch of $y = \sqrt{1+e^x}$.

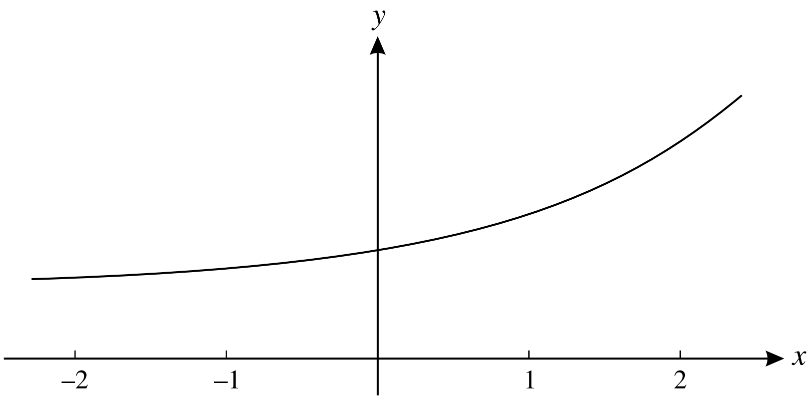


**Fig. 1**

**(ii)** Suppose that the trapezium rule is used with more strips than in part **(i)** to estimate $\int_{-2}^{2} \sqrt{1+e^x}\, dx$. State, with a reason but no further calculation, whether this would give a larger or smaller estimate. **[2]**

**2** A curve is defined parametrically by the equations

$$x = \frac{1}{1+t}, \qquad y = \frac{1-t}{1+2t}.$$

Find $t$ in terms of $x$. Hence find the cartesian equation of the curve, giving your answer as simply as possible. **[5]**

**3** Find the first three terms in the binomial expansion of $\frac{1}{(3-2x)^3}$ in ascending powers of $x$. State the set of values of $x$ for which the expansion is valid. **[7]**

**4** The points A, B and C have coordinates (2, 0, −1), (4, 3, −6) and (9, 3, −4) respectively.

**(i)** Show that AB is perpendicular to BC. **[4]**

**(ii)** Find the area of triangle ABC. **[3]**

**5** Show that $\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$. **[3]**

**6** **(i)** Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} -8 \\ -2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ and the plane $2x - 3y + z = 11$. **[4]**

**(ii)** Find the acute angle between the line and the normal to the plane. **[4]**

**Section B** (36 marks)

**7** A particle is moving vertically downwards in a liquid. Initially its velocity is zero, and after $t$ seconds it is $v\,\mathrm{m\,s^{-1}}$. Its terminal (long-term) velocity is $5\,\mathrm{m\,s^{-1}}$.

A model of the particle's motion is proposed. In this model, $v = 5\left(1 - \mathrm{e}^{-2t}\right)$.

**(i)** Show that this equation is consistent with the initial and terminal velocities. Calculate the velocity after 0.5 seconds as given by this model. **[3]**

**(ii)** Verify that $v$ satisfies the differential equation $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 10 - 2v$. **[3]**

In a second model, $v$ satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.4v^2.$$

As before, when $t = 0$, $v = 0$.

**(iii)** Show that this differential equation may be written as

$$\frac{10}{(5 - v)(5 + v)} \frac{\mathrm{d}v}{\mathrm{d}t} = 4.$$

Using partial fractions, solve this differential equation to show that

$$t = \tfrac{1}{4} \ln\left(\frac{5 + v}{5 - v}\right).$$ **[8]**

This can be re-arranged to give $v = \dfrac{5\left(1 - \mathrm{e}^{-4t}\right)}{1 + \mathrm{e}^{-4t}}$. [You are **not** required to show this result.]

**(iv)** Verify that this model also gives a terminal velocity of $5\,\mathrm{m\,s^{-1}}$.

Calculate the velocity after 0.5 seconds as given by this model. **[3]**

The velocity of the particle after 0.5 seconds is measured as $3\,\mathrm{m\,s^{-1}}$.

**(v)** Which of the two models fits the data better? **[1]**

**8** Fig. 8 shows a searchlight, mounted at a point A, 5 metres above level ground. Its beam is in the shape of a cone with axis AC, where C is on the ground. AC is angled at $\alpha$ to the vertical. The beam produces an oval-shaped area of light on the ground, of length DE. The width of the oval at C is GF. Angles DAC, EAC, FAC and GAC are all $\beta$.

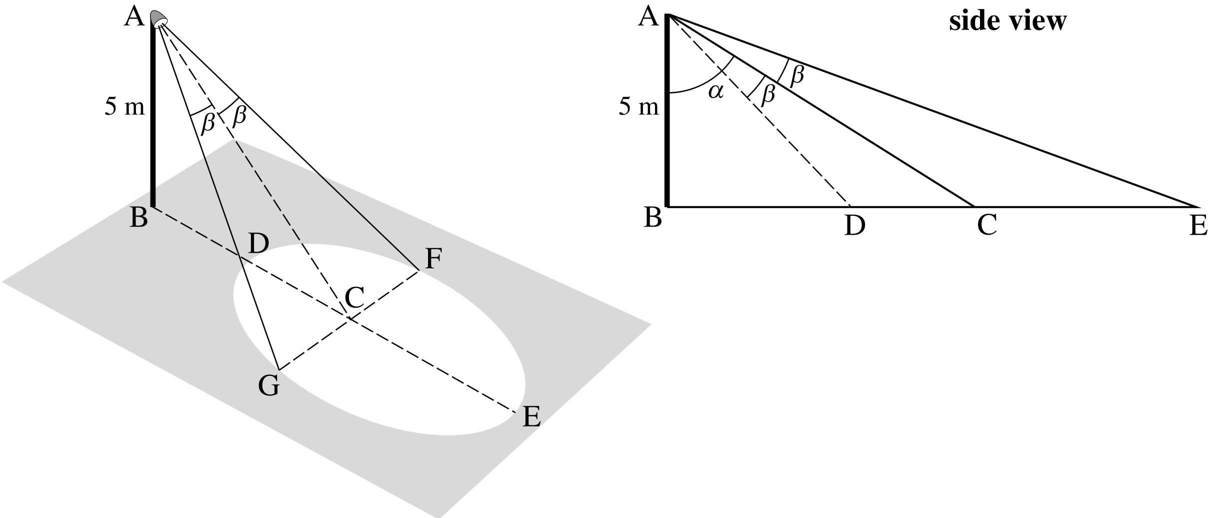


**Fig. 8**

In the following, all lengths are in metres.

**(i)** Find AC in terms of $\alpha$, and hence show that $\text{GF} = 10\sec\alpha\tan\beta$. **[3]**

**(ii)** Show that $\text{CE} = 5\big(\tan(\alpha+\beta) - \tan\alpha\big)$.

Hence show that $\text{CE} = \dfrac{5\tan\beta\sec^2\alpha}{1-\tan\alpha\tan\beta}$. **[5]**

Similarly, it can be shown that $\text{CD} = \dfrac{5\tan\beta\sec^2\alpha}{1+\tan\alpha\tan\beta}$. [You are **not** required to derive this result.]

You are now given that $\alpha = 45^\circ$ and that $\tan\beta = t$.

**(iii)** Find CE and CD in terms of $t$. Hence show that $\text{DE} = \dfrac{20t}{1-t^2}$. **[5]**

**(iv)** Show that $\text{GF} = 10\sqrt{2}\,t$. **[2]**

For a certain value of $\beta$, DE = 2GF.

**(v)** Show that $t^2 = 1 - \dfrac{1}{\sqrt{2}}$.

Hence find this value of $\beta$. **[3]**

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**ADVANCED GCE**

**MATHEMATICS (MEI)** **4754B**

Applications of Advanced Mathematics (C4) Paper B: Comprehension

| Candidates answer on the question paper. |
|---|
| **OCR supplied materials:** |
| • Insert (inserted) |
| • MEI Examination Formulae and Tables (MF2) |
| **Other materials required:** |
| • Scientific or graphical calculator |
| • Rough paper |

**Friday 14 January 2011**
**Afternoon**

**Duration:** Up to 1 hour

| Candidate forename | | Candidate surname | |
|---|---|---|---|

| Centre number | | | | | | Candidate number | | | | |
|---|---|---|---|---|---|---|---|---|---|---|

**INSTRUCTIONS TO CANDIDATES**

- The insert will be found in the centre of this document.
- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Write your answer to each question in the space provided. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- The insert contains the text for use with the questions.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **18**.
- This document consists of **8** pages. Any blank pages are indicated.

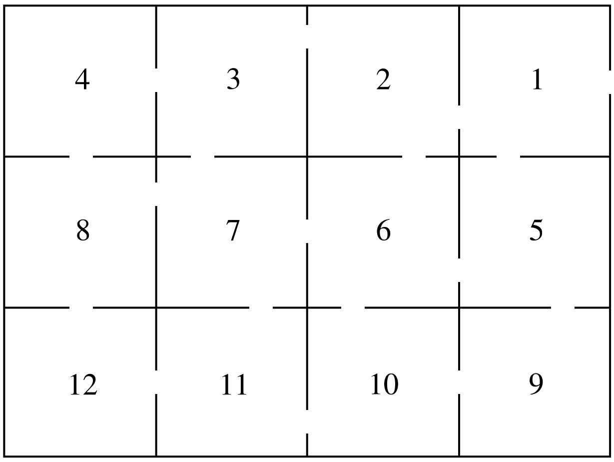
**INSTRUCTION TO EXAMS OFFICER / INVIGILATOR**

- This paper should be attached to the candidate's paper A script before sending to the examiner.

| Examiner's Use Only: | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| **Total** | |

For Examiner's Use

**1** The gallery shown below is a 3 by 4 grid of rectangular rooms.




Entrance and Exit

**(i)** Mark on the diagram the positions of six guards so that the whole gallery can be observed. **[1]**

**(ii)** Give a counterexample to disprove the following proposition:

For an $m$ by $n$ grid of rectangular rooms, $\left\lfloor \frac{mn}{2} \right\rfloor$ guards are required. **[2]**

For Examiner's Use

**2** **(i)** Show that $\dfrac{(2r+1)-(-1)^r}{4} = \left\lfloor \dfrac{r+1}{2} \right\rfloor$ in the case where $r = 4$. **[2]**

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**(ii)** The ceiling function, $\lceil x \rceil$, is defined as the smallest integer greater than or equal to $x$.

Complete the following table. **[2]**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\left\lceil \dfrac{x}{2} \right\rceil$ | | | | | |

**3** Justify the statement in lines 79 and 80. **[2]**

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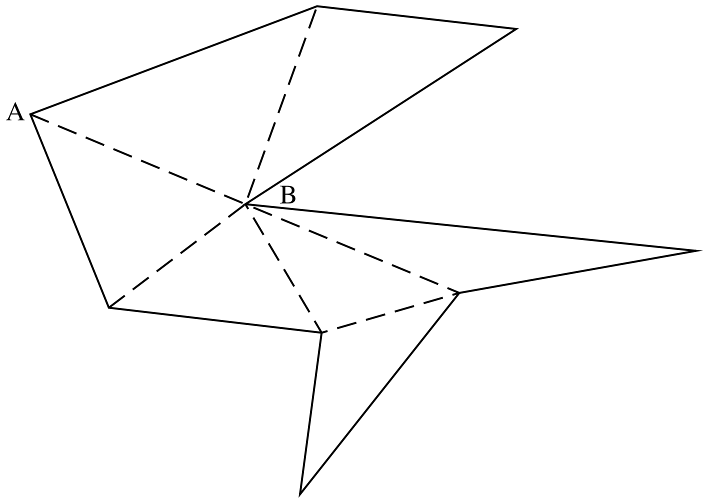
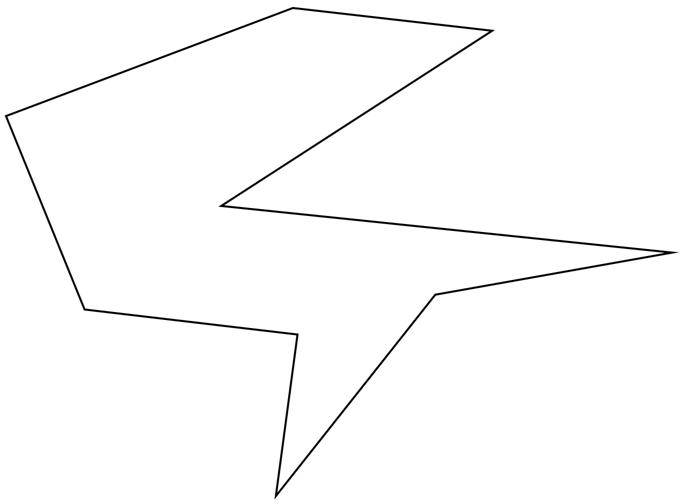
For Examiner's Use

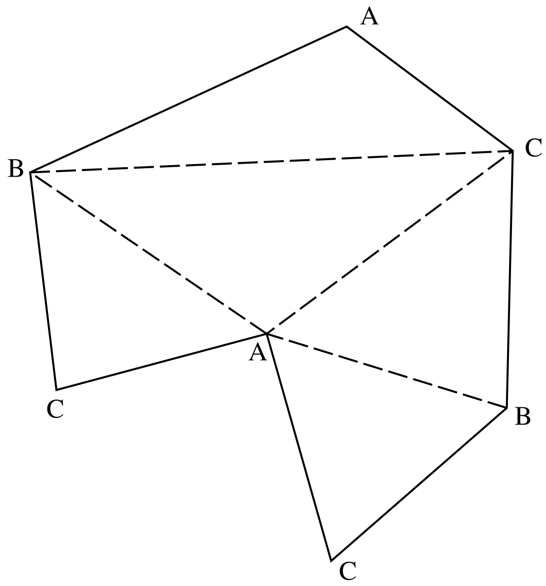
**4** **(i)** Following the procedure in Fig. 6, complete the labelling of the polygon shown below. **[2]**




**(ii)** In order to use the minimum number of cameras, show on the diagram below where your answer to part **(i)** indicates the cameras should be placed. **[1]**



**5** With reference to the labelled triangulation shown below, state with a reason whether each of the following statements is true or false.




**(i)** The triangulation shows that 2 cameras are sufficient. **[2]**

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Reason: ............................................................................................................................

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**(ii)** The triangulation shows that 2 cameras are necessary. **[2]**

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Reason: ............................................................................................................................

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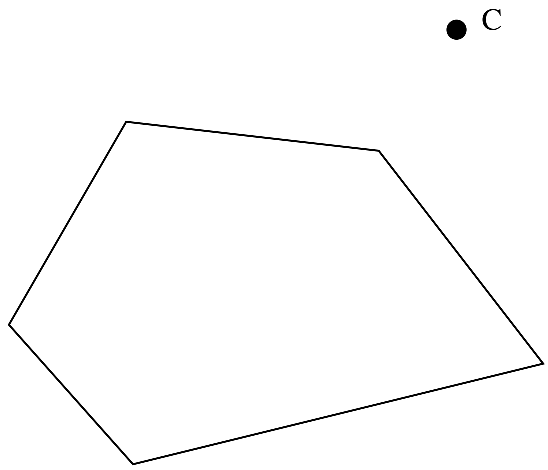
For Examiner's Use

**6** On lines 96 and 97 it says "If the cameras did not need to be mounted on the walls, but could be positioned further away from the building, then fewer cameras would usually suffice."

Using the diagram below, which shows a pentagon and one external camera, C, indicate by shading the region in which a second camera must be positioned so that all the walls could be observed by the two cameras. **[2]**



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**ADVANCED GCE**

# MATHEMATICS (MEI) 4754B

Applications of Advanced Mathematics (C4) Paper B: Comprehension

**INSERT**

**Friday 14 January 2011**
**Afternoon**

**Duration:** Up to 1 hour

*4754B*

**INFORMATION FOR CANDIDATES**

- This insert contains the text for use with the questions.
- This document consists of **12** pages. Any blank pages are indicated.

**INSTRUCTION TO EXAMS OFFICER / INVIGILATOR**

- Do not send this insert for marking; it should be retained in the centre or destroyed.

# The Art Gallery Problem

## Introduction

Closed-circuit television (CCTV) is widely used to monitor activities in public areas. Usually a CCTV camera is fixed to a wall, either on the inside or the outside of a building, in such a way that it can rotate to survey a wide region.

Art galleries need to take surveillance very seriously. Many galleries use a combination of guards, who can move between rooms, and fixed cameras.

This article addresses the problem of how to ensure that all points in a gallery can be observed, using the minimum number of guards or cameras. Two typical layouts will be considered: a standard layout consisting of a chain of rectangular rooms with one route through them; and a polygonal, open-plan gallery.

## Standard layout

Fig. 1 shows the plan view of an art gallery. It contains six rectangular rooms in a chain.

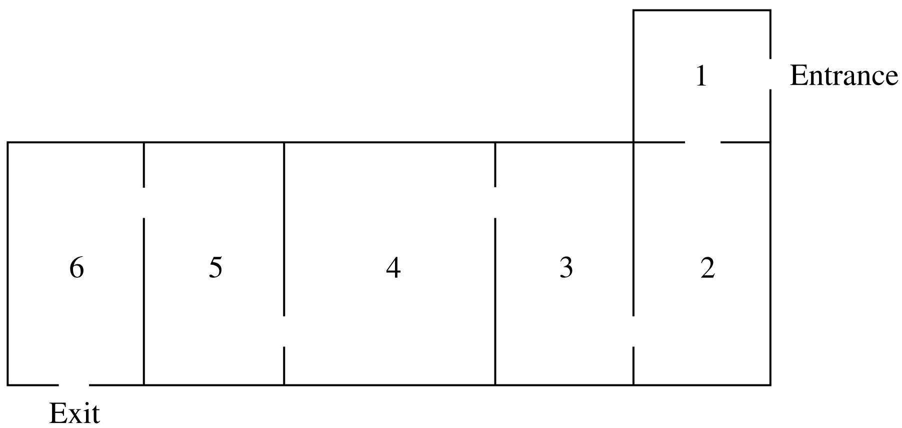


**Fig. 1**

Imagine you want to employ the minimum number of guards so that every point in the gallery can be observed by at least one of them. How many guards are needed and where would you position them?

It turns out that 3 guards are needed; they should be positioned in the doorways between rooms 1 and 2, rooms 3 and 4 and rooms 5 and 6.

Table 2 shows the number of guards, $G$, needed for different numbers of rectangular rooms, $r$, arranged in a chain.

| Number of rooms, $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of guards, $G$ | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |

**Table 2**

One formula which expresses $G$ in terms of $r$ is

$$G = \frac{(2r+1) - (-1)^r}{4}.$$

This can be expressed more concisely using the 'floor function', denoted by $\lfloor x \rfloor$. This is defined as the greatest integer less than or equal to $x$. For example, $\lfloor 3.9 \rfloor = 3$ and $\lfloor 5 \rfloor = 5$.

Using this notation, the formula which expresses $G$ in terms of $r$ is

$$G = \left\lfloor \frac{r+1}{2} \right\rfloor.$$

It has been assumed that the thickness of the walls does not obscure the view of a guard positioned in a doorway. In reality the guard would need to take a step into a room to ensure that he or she can see into all corners. In this sense, guards have the advantage over fixed cameras as the thickness of the walls between rooms would result in the view of a camera positioned in a doorway being partially blocked.

The remainder of this article is concerned with positioning fixed cameras in galleries which are not divided into separate rooms.

**Open-plan gallery**

An open-plan gallery has no interior walls.

Figs 3a, 3b, 3c and 3d show the plan views of four different open-plan galleries. Each one is made up of 12 straight exterior walls with no interior walls.

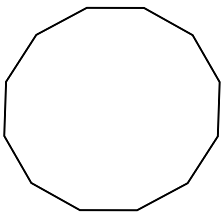

**Fig. 3a**

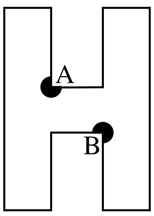


**Fig. 3b**

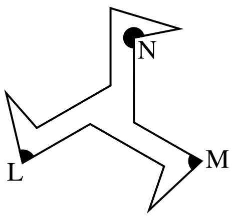


**Fig. 3c**

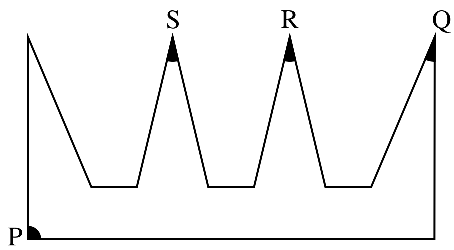


**Fig. 3d**

Fig. 3a is a gallery in the shape of a convex dodecagon. A single camera mounted at any point on any wall would be able to observe the entire gallery.

In order to ensure that all points can be observed, it turns out that the galleries shown in Figs 3b, 3c and 3d require 2, 3 and 4 cameras respectively. Possible positions for the cameras are shown. The cameras do not have to be positioned in corners but corners are often convenient locations for them.

An interesting question is whether there could be an open-plan gallery with 12 straight walls that requires more than 4 cameras.

A procedure called *triangulation* helps to answer this question and to determine how many cameras may be required for open-plan galleries with any number of walls. This will be illustrated for a gallery with 11 walls.

**Triangulation**

Fig. 4 shows a polygon with 11 edges.

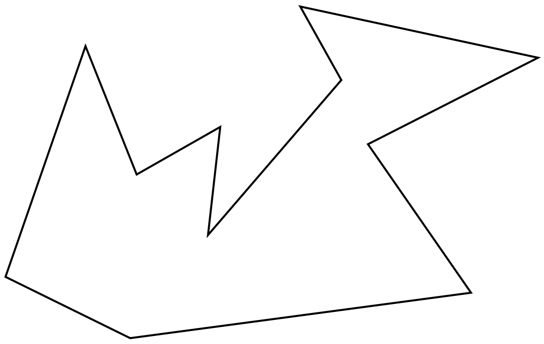

**Fig. 4**

It can be proved that every polygon can be split into triangles without creating any new vertices; this procedure is called triangulation. Figs 5a and 5b show two ways of triangulating the polygon shown in Fig. 4.

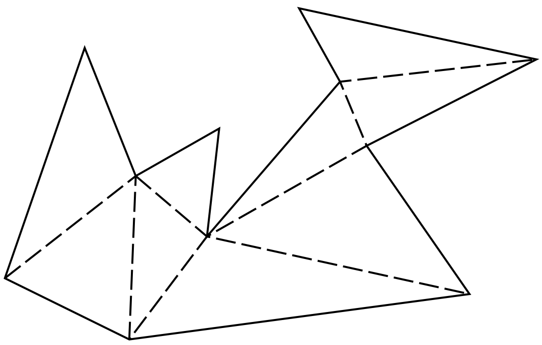

**Fig. 5a**

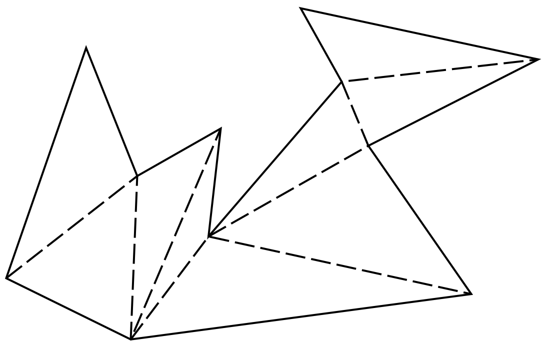

**Fig. 5b**

There is a two-stage process to decide on possible positions of the cameras in an open-plan gallery. The first stage is to add new internal edges to triangulate the polygon which represents the gallery. Then each vertex is labelled either A, B or C using the procedure shown in Fig. 6.

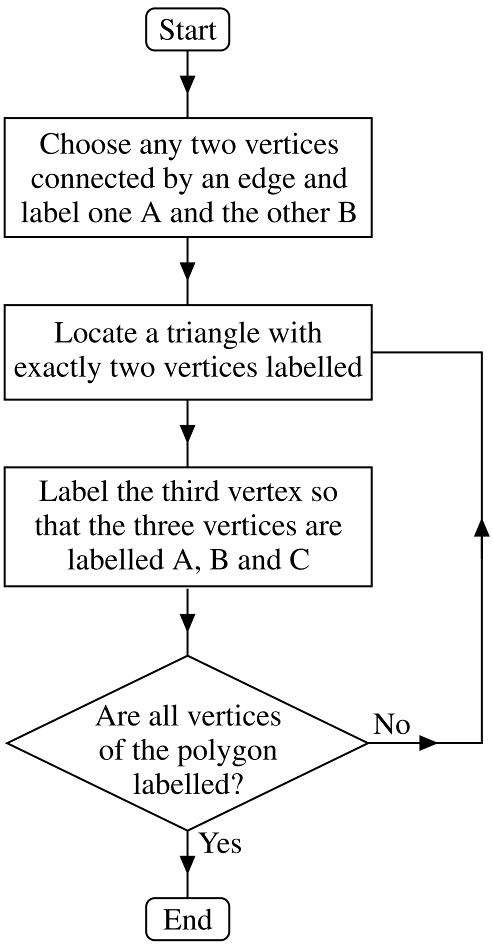


**Fig. 6**

This procedure is illustrated using the polygon in Fig. 4; for ease of reference the polygon has been reproduced in Fig. 7 with the vertices numbered 1 to 11.

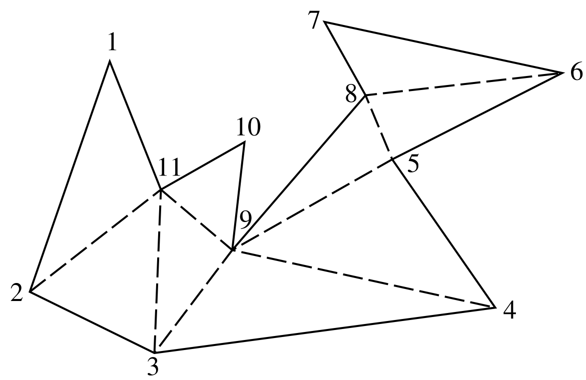


**Fig. 7**

Fig. 8 shows the result after choosing vertices 1 and 2 and labelling them A and B respectively.

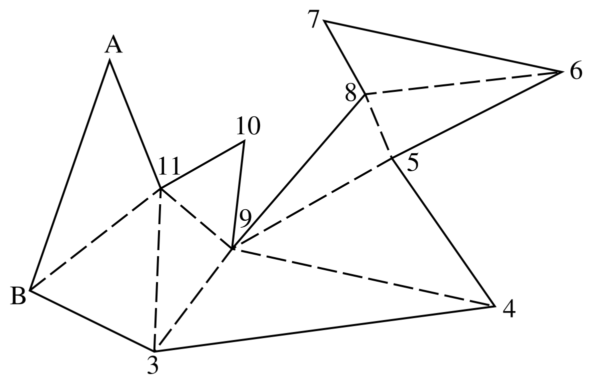


**Fig. 8**

Vertex 11 is now assigned the label C so that this triangle (shaded in Fig. 9) has vertices labelled A, B and C.

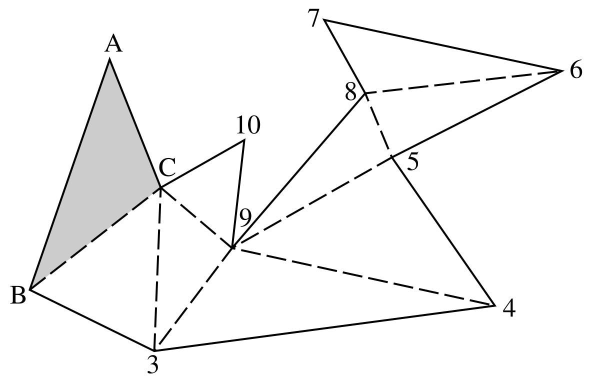


**Fig. 9**

The next vertex to be labelled is vertex 3; this is assigned the label A. The remaining vertices are labelled in the following order.

Vertex 9 is labelled B;
Vertex 10 is labelled A;
Vertex 4 is labelled C;
Vertex 5 is labelled A;
Vertex 8 is labelled C;
Vertex 6 is labelled B;
Vertex 7 is labelled A.

The resulting labelling is shown in Fig. 10.

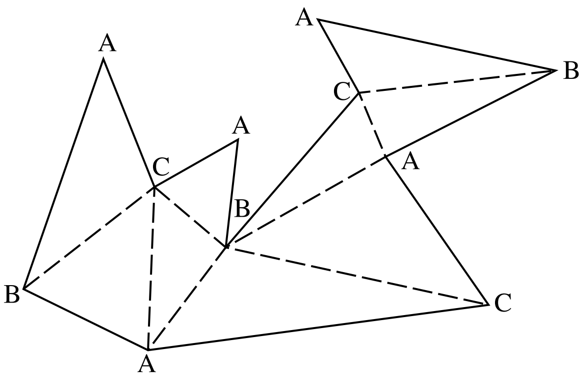


**Fig. 10**

The 11 vertices have all been assigned a label A, B or C; the numbers of vertices with each label is given in Table 11.

| Label | Number of vertices assigned this label |
|---|---|
| A | 5 |
| B | 3 |
| C | 3 |

**Table 11**

Since every triangle contains a vertex labelled A, positioning a camera at each vertex A will ensure that the whole gallery can be observed. This would require 5 cameras.

Alternatively, positioning cameras at the 3 vertices labelled B would ensure that the whole gallery can be observed using only 3 cameras.

Positioning cameras at the 3 vertices labelled C would also be sufficient.

Since $\frac{11}{3} < 4$, in any 11-sided polygon at least one of A, B or C must appear as a label at most 3 times.

A generalisation of this argument demonstrates that an open-plan gallery with $n$ walls can be covered with $\left\lfloor \frac{n}{3} \right\rfloor$ cameras or fewer. Table 12 shows this information.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\left\lfloor \frac{n}{3} \right\rfloor$ | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 |

**Table 12**

It is always possible to design an open-plan gallery with $n$ walls that requires $\left\lfloor \frac{n}{3} \right\rfloor$ cameras.

**Triangulation in practice**

For a given open-plan gallery, different triangulations may suggest different numbers of cameras.

Figs 13a and 13b show two different triangulations on a particular hexagon. Fig. 13a shows that the whole gallery can be observed using 2 cameras (as in Table 12). However, Fig. 13b shows that only one camera is necessary.

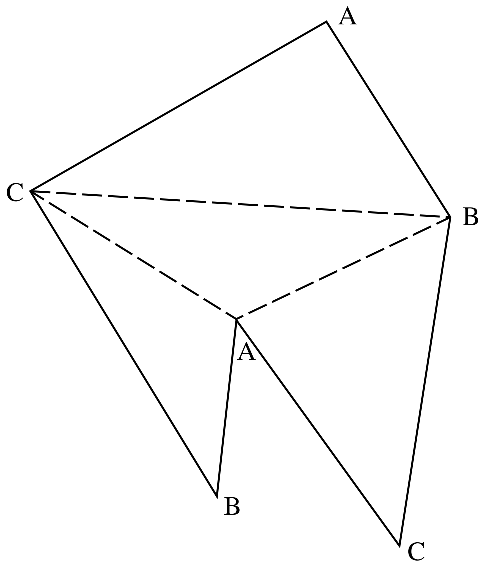


**Fig. 13a**

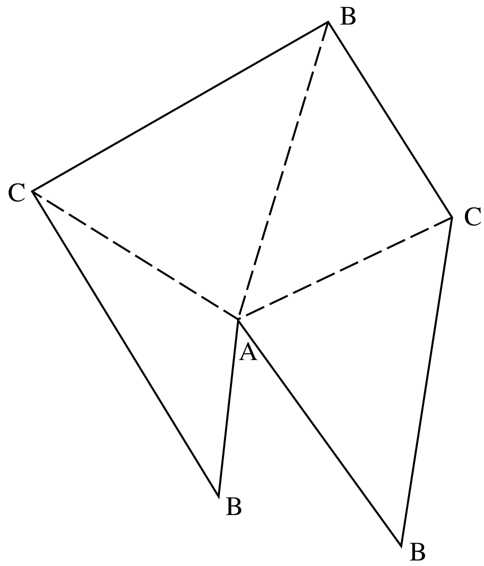


**Fig. 13b**

**Surveillance of the outside of a building**

Fig. 14 shows a pentagonal building with corners numbered, in order, from 1 to 5. To observe all of the outside of this building, 3 cameras could be positioned at the odd-numbered corners as shown.

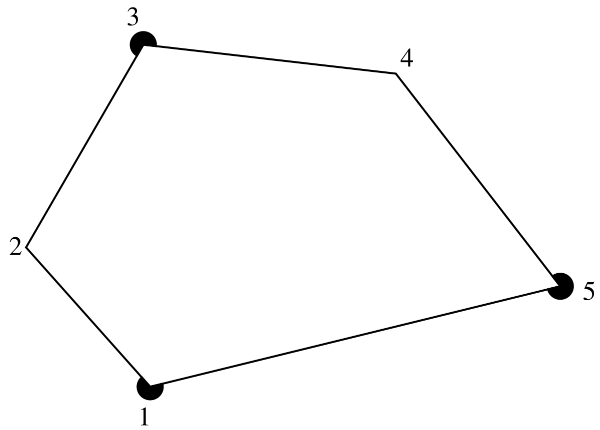


**Fig. 14**

This method of positioning cameras at odd-numbered corners can be extended to a polygonal building with any number of walls, showing that $\left\lfloor \frac{n+1}{2} \right\rfloor$ cameras are sufficient to observe all the outside of any $n$-sided polygonal building.

If the cameras did not need to be mounted on the walls, but could be positioned further away from the building, then fewer cameras would usually suffice.

## Conclusion

Triangulation provides an elegant proof when analysing the minimum number of cameras needed in open-plan galleries. With more complex layouts in two and three dimensions, such elegant solutions have not been discovered although some necessary and some sufficient conditions have been found. In general, optimal solutions are found by applying computer algorithms to mathematical models of galleries.