Oxford Cambridge and RSA

Monday 18 October 2021 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

8980673136

You must have:
- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \frac{u}{v}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

Integration

$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$

$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,\mathrm{d}x = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$

Integration by parts $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \mathrm{si}\ \ A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k+\frac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r\, p^r q^{n-r}$ where $q = 1 - p$

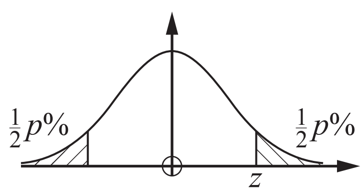
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 **(a)** Express x^2+8x+2 in the form $(x+a)^2+b$. **[2]**

(b) Write down the coordinates of the turning point of the curve $y=x^2+8x+2$. **[1]**

(c) State the transformation(s) which map(s) the curve $y=x^2$ onto the curve $y=x^2+8x+2$. **[2]**

2 Solve the equation $\sin 2x = 0.3$ for $0° \leqslant x \leqslant 180°$. Give your answer(s) correct to **1** decimal place. **[2]**

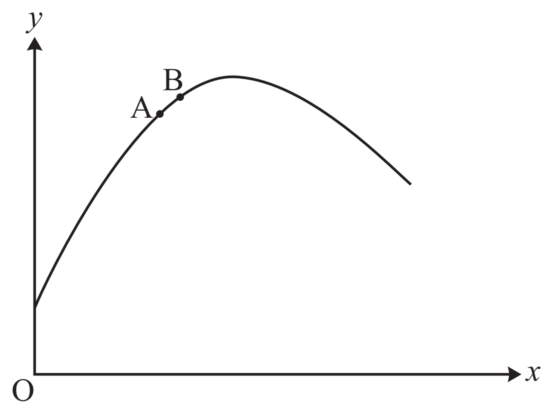
3 **(a)** Determine, in terms of k, the coordinates of the point where the lines with the following equations intersect.

$$x+y=k$$
$$2x-y=1$$ **[3]**

(b) Determine, in terms of k, the coordinates of the points where the line $x+y=k$ crosses the curve $y=x^2+k$. **[4]**

4 The diagram shows points A and B on the curve $y=\left(\frac{x}{4}\right)^{-x}$.

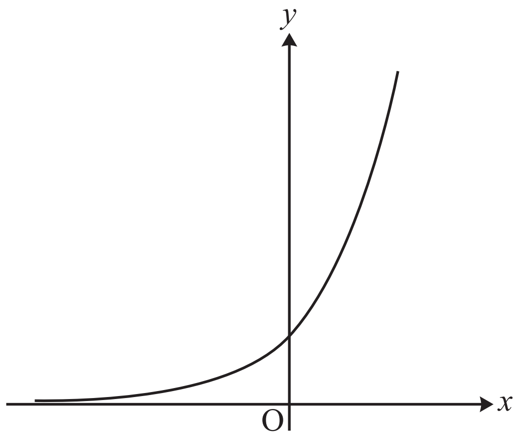
The x-coordinate of A is 1 and the x-coordinate of B is 1.1.



(a) Find the gradient of chord AB. Give your answer correct to **2** decimal places. **[2]**

(b) Give the x-coordinate of a point C on the curve such that the gradient of chord AC is a better approximation to the gradient of the tangent to the curve at A. **[1]**

5 **(a)** The diagram shows the curve $y = e^x$.



On the axes in the Printed Answer Booklet, sketch graphs of

(i) $\frac{dy}{dx}$ against x, **[1]**

(ii) $\frac{dy}{dx}$ against y. **[2]**

(b) Wolves were introduced to Yellowstone National Park in 1995.

The population of wolves, y, is modelled by the equation

$y = Ae^{kt}$,

where A and k are constants and t is the number of years after 1995.

(i) Give a reason why this model might be suitable for the population of wolves. **[1]**

(ii) When $t = 0$, $y = 21$ and when $t = 1$, $y = 51$.

Find values of A and k consistent with the data. **[3]**

(iii) Give a reason why the model will not be a good predictor of wolf populations many years after 1995. **[1]**

6 **In this question you must show detailed reasoning.**

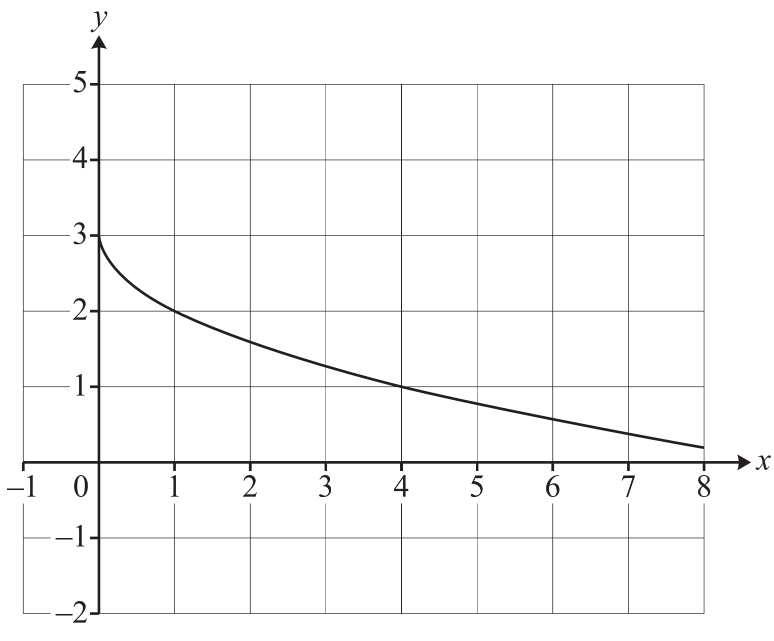
Show that $\sum_{r=1}^{3} \frac{1}{\sqrt{r+1}+\sqrt{r}} = 1$. **[4]**

7 Determine $\int x \cos 2x \quad x$. **[3]**

8 For a particular value of a, the curve $y = \frac{a}{x^2}$ passes through the point (3, 1).

Find the coordinates of all the other points on the curve where both the x-coordinate and the y-coordinate are integers. **[3]**

9 The diagram shows the curve $y = 3 - \sqrt{x}$.



(a) Draw the line $y = 5x - 1$ on the copy of the diagram in the Printed Answer Booklet. **[1]**

(b) **In this question you must show detailed reasoning.**

Determine the exact area of the region bounded by the curve $y = 3 - \sqrt{x}$, the lines $y = 5x - 1$ and $x = 4$ and the x-axis. **[10]**

10 **(a)** Express $\dfrac{1}{(4x+1)(x+1)}$ in partial fractions. **[3]**

(b) A curve passes through the point (0, 2) and satisfies the differential equation

$$\frac{dy}{dx} = \frac{y}{(4x+1)(x+1)},$$

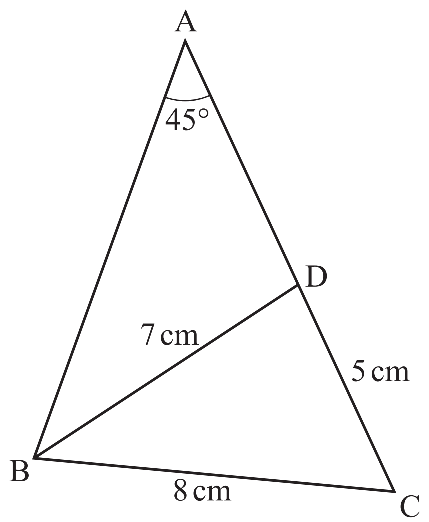
for $x > -\frac{1}{4}$.

Show by integration that $y = A\left(\dfrac{4x+1}{x+1}\right)^B$ where A and B are constants to be determined. **[6]**

11 **In this question you must show detailed reasoning.**

The diagram shows triangle ABC, with BC = 8 cm and angle BAC = 45°.

The point D on AC is such that DC = 5 cm and BD = 7 cm.



Determine the exact length of AB. **[5]**

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

12 Show that $\beta = \arctan\left(\frac{1}{3}\right)$, as given in line 15. **[3]**

13 **(a)** Use triangle ABE in **Fig. C2** to show that $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$, as given in line 29. **[1]**

(b) Sketch the graph of $y = \arctan x$. **[1]**

(c) What property of the arctan function ensures that $y > \frac{1}{x} \Rightarrow \arctan y > \arctan\left(\frac{1}{x}\right)$, as given in line 30? **[1]**

14 **(a)** Show that

$$\arctan\left(\frac{1}{n+1}\right) + \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right) \Rightarrow \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1.$$ **[1]**

(b) Use the arctan addition formula in line 23 to show that

$$\arctan\left(\frac{1}{n+1}\right) + \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right),$$ as given in line 39. **[4]**

15 Prove that $\arctan 1 + \arctan 2 + \arctan 3 = \pi$, as given in line 41. **[4]**

END OF QUESTION PAPER

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Insert

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- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

- This Insert contains the article for Section B.
- This document has **4** pages.

Adding arctangents

Where does the name 'arctangent' come from?

The two commonly used ways to denote the angle which has a tangent x are $\tan^{-1}x$ and $\arctan x$. The first of these is related to inverse function notation, $\mathrm{f}^{-1}(x)$. Arctangent comes from radian measure, where an angle is represented by an arc on a unit circle; $\arctan x$ is the arc whose tangent is x.

An interesting result

It can be shown that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right) = \arctan 1$.

Consider the diagram in **Fig. C1**.

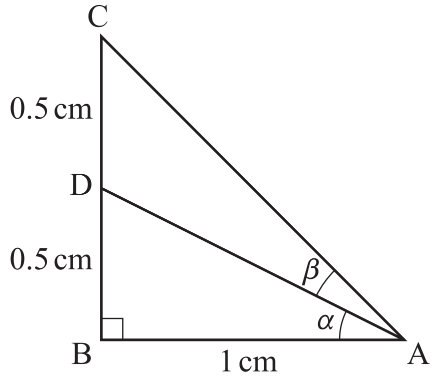

Fig. C1

Triangle ABC is right-angled at B.
AB = BC = 1 cm.
D is the midpoint of BC.

Using triangle ABD, $\tan\alpha = \frac{\text{DB}}{\text{BA}} = \frac{1}{2}$ so $\alpha = \arctan\left(\frac{1}{2}\right)$.

Using triangle ABC, $\tan(\alpha+\beta) = 1$ so $\alpha+\beta = \arctan 1$.

Hence $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = 1$.

Using $\tan\alpha = \frac{1}{2}$ and finding $\tan\beta$, it follows that $\beta = \arctan\left(\frac{1}{3}\right)$,

which gives the required result that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right) = \arctan 1$.

Generalising the result

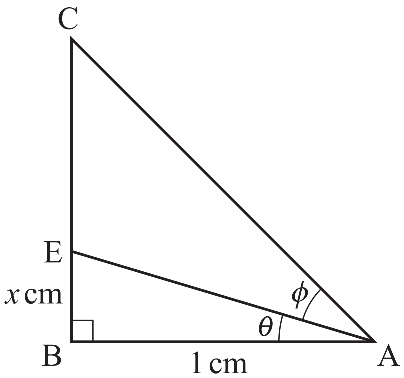

Fig. C2

Triangle ABC in **Fig. C2** is the same as triangle ABC in **Fig. C1** but E is a point on BC such that EB = x cm and $\theta = \arctan x$.

Following the same method as above, $\arctan x + \arctan\left(\frac{1-x}{1+x}\right) = \arctan 1$.

The arctan addition formula

The arctangent addition formula is a further generalization:

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$, as long as $xy < 1$.

This result is equivalent to the addition formula $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ where $\alpha = \arctan x$ and $\beta = \arctan y$.

To see why the restriction $xy < 1$ is necessary, consider what happens if $xy \geqslant 1$.

Clearly, $\frac{x+y}{1-xy}$ is undefined when $xy = 1$, so the formula does not apply in this case.

Suppose next that $xy > 1$, and that x and y are both positive; in this case $y > \frac{1}{x}$.

For any positive x, $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$.

$y > \frac{1}{x} \Rightarrow \arctan y > \arctan\left(\frac{1}{x}\right)$ so it follows that $\arctan x + \arctan y > \frac{\pi}{2}$.

However, $\arctan\left(\frac{x+y}{1-xy}\right)$ cannot be greater than $\frac{\pi}{2}$ as the range of the arctan function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The formula $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ therefore cannot be valid in this case.

A similar argument can be used to show that the formula cannot be valid when $xy > 1$ and x and y are both negative.

If $xy > 1$, the arctangent addition formula needs to be adapted, as shown below.

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi$, when $xy > 1$ and $x, y < 0$

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$, when $xy > 1$ and $x, y > 0$

Some additional results

- For n a positive integer, $\arctan\left(\frac{1}{n+1}\right) + \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right)$; this follows directly from the arctan addition formula in line 23.
- $\arctan 1 + \arctan 2 + \arctan 3 = \pi$. This can be proved by using $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ together with $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1$.

