OCR
RECOGNISING ACHIEVEMENT

Tuesday 17 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4754A Applications of Advanced Mathematics (C4) Paper A

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4754A
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

4733130112

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.
- This paper will be followed by **Paper B: Comprehension**.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this Question Paper for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Section A (36 marks)

1 Express $\dfrac{x+1}{x^2(2x-1)}$ in partial fractions. **[5]**

2 Solve, correct to 2 decimal places, the equation $\cot 2\theta = 3$ for $0° \leqslant \theta \leqslant 180°$. **[4]**

3 Express $3\sin x + 2\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence find, correct to 2 decimal places, the coordinates of the maximum point on the curve $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = 3\sin x + 2\cos x, \quad 0 \leqslant x \leqslant \pi.$$ **[7]**

4 **(i)** Complete the table of values for the curve $y = \sqrt{\cos x}$.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y		0.9612	0.8409		

Hence use the trapezium rule with strip width $h = \frac{\pi}{8}$ to estimate the value of the integral $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, \mathrm{d}x$, giving your answer to 3 decimal places. **[3]**

Fig. 4 shows the curve $y = \sqrt{\cos x}$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

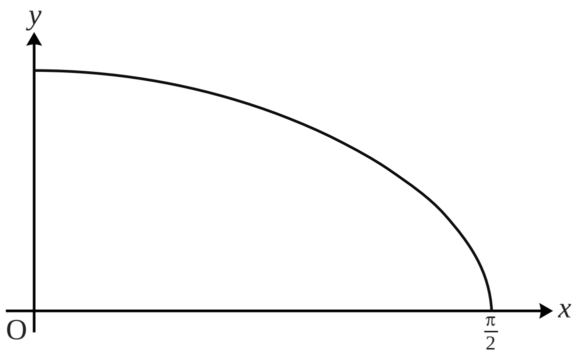


Fig. 4

(ii) State, with a reason, whether the trapezium rule with a strip width of $\frac{\pi}{16}$ would give a larger or smaller estimate of the integral. **[1]**

5 Verify that the vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ is perpendicular to the plane through the points A(2, 0, 1), B(1, 2, 2) and C(0, –4, 1). Hence find the cartesian equation of the plane. **[5]**

6 Given the binomial expansion $(1 + qx)^p = 1 - x + 2x^2 + \dots$, find the values of p and q. Hence state the set of values of x for which the expansion is valid. **[6]**

7 Show that the straight lines with equations $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ meet.

Find their point of intersection. **[5]**

Section B (36 marks)

8 Fig. 8 shows a cross-section of a car headlight whose inside reflective surface is modelled, in suitable units, by the curve

$$x = 2t^2,\ y = 4t, \quad -\sqrt{2} \leqslant t \leqslant \sqrt{2}\ .$$

P$(2t^2, 4t)$ is a point on the curve with parameter t. TS is the tangent to the curve at P, and PR is the line through P parallel to the x-axis. Q is the point (2, 0). The angles that PS and QP make with the positive x-direction are θ and ϕ respectively.

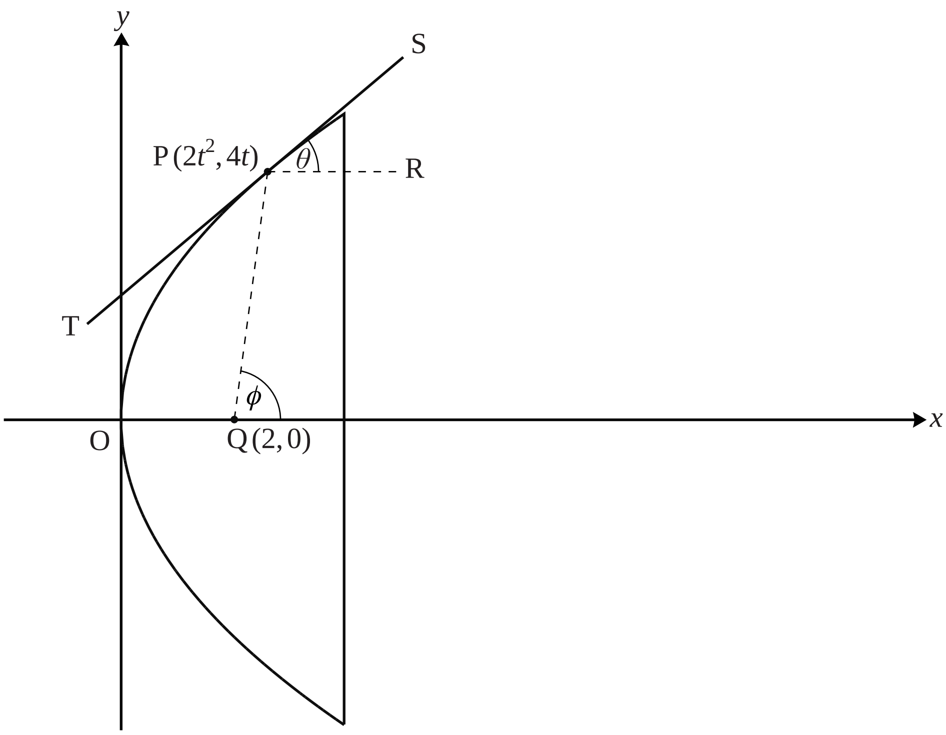


Fig. 8

(i) By considering the gradient of the tangent TS, show that $\tan\theta = \frac{1}{t}$. **[3]**

(ii) Find the gradient of the line QP in terms of t. Hence show that $\phi = 2\theta$, and that angle TPQ is equal to θ. **[8]**

[The above result shows that if a lamp bulb is placed at Q, then the light from the bulb is reflected to produce a parallel beam of light.]

The inside surface of the headlight has the shape produced by rotating the curve about the x-axis.

(iii) Show that the curve has cartesian equation $y^2 = 8x$. Hence find the volume of revolution of the curve, giving your answer as a multiple of π. **[7]**

9

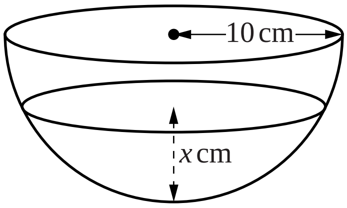


Fig. 9

Fig. 9 shows a hemispherical bowl, of radius 10 cm, filled with water to a depth of x cm. It can be shown that the volume of water, V cm^3, is given by

$$V = \pi(10x^2 - \tfrac{1}{3}x^3).$$

Water is poured into a leaking hemispherical bowl of radius 10 cm. Initially, the bowl is empty. After t seconds, the volume of water is changing at a rate, in cm^3 s^{-1}, given by the equation

$$\frac{dV}{dt} = k(20 - x),$$

where k is a constant.

(i) Find $\dfrac{dV}{dx}$, and hence show that $\pi x \dfrac{dx}{dt} = k$. **[4]**

(ii) Solve this differential equation, and hence show that the bowl fills completely after T seconds, where $T = \dfrac{50\pi}{k}$. **[5]**

Once the bowl is full, the supply of water to the bowl is switched off, and water then leaks out at a rate of kx cm^3 s^{-1}.

(iii) Show that, t seconds later, $\pi(20 - x)\dfrac{dx}{dt} = -k$. **[3]**

(iv) Solve this differential equation.

Hence show that the bowl empties in $3T$ seconds. **[6]**

OCR
RECOGNISING ACHIEVEMENT

Tuesday 17 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4754B Applications of Advanced Mathematics (C4) Paper B: Comprehension

QUESTION PAPER

Candidates answer on the Question Paper.

OCR supplied materials:
- Insert (inserted)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator
- Rough paper

Duration: Up to 1 hour

Candidate forename		Candidate surname	

Centre number						Candidate number				

INSTRUCTIONS TO CANDIDATES

- The Insert will be found in the centre of this document.
- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Write your answer to each question in the space provided. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- The insert contains the text for use with the questions.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **18**.
- This document consists of **8** pages. Any blank pages are indicated.

DC (NH/JG) 44282/3

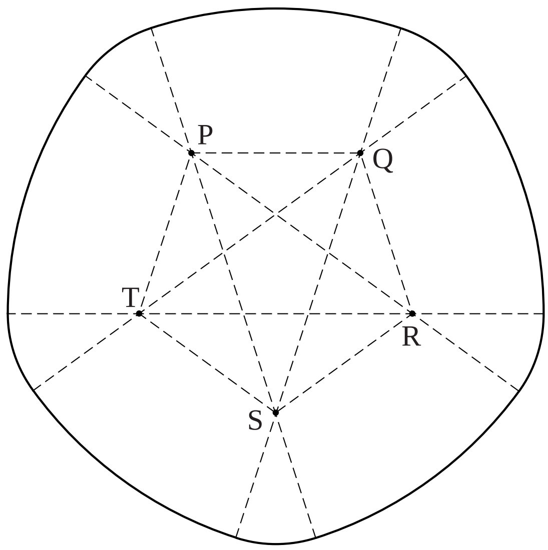
1 In lines 22 and 23 it says "arcs can be added to any regular polygon with an odd number of sides to make a curve of constant width". State why the method described cannot be applied to a regular polygon with an even number of sides. **[1]**

1	

2 **(i)** On the curve of constant width below, indicate clearly the arcs that were constructed with centre P. **[1]**

(ii) Given that this curve has perimeter 70 cm, calculate its width, correct to 3 significant figures. **[2]**

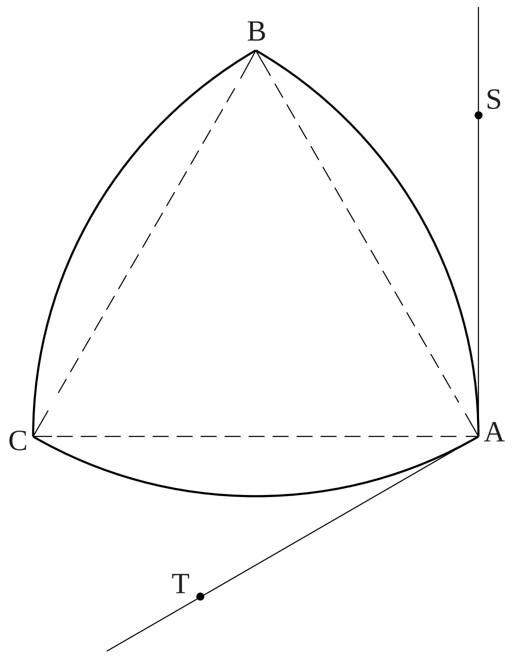
2 (i)



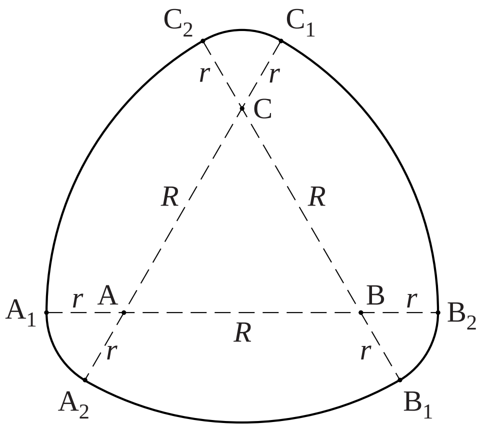
2 (ii)	

3 The diagram below shows two tangents, AS and AT, at vertex A on a Reuleaux triangle. State the angle SAT justifying your answer carefully. **[3]**

3



4 For the curve in Fig. 7b (copied below) the width, l, is $R + 2r$.



(i) Prove that the perimeter is πl. **[3]**

(ii) You are given that, in the case where $r = \frac{R}{2}$, the area enclosed by this curve is $R^2\left(\frac{5\pi - 2\sqrt{3}}{4}\right)$.

Show that this area falls in the range indicated in lines 28 and 29. **[3]**

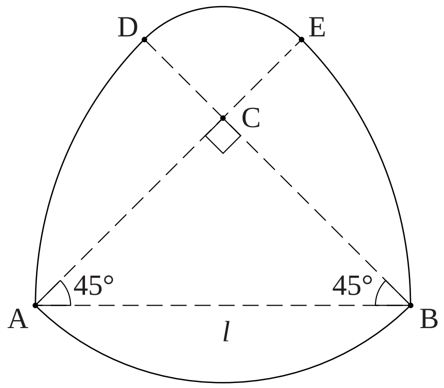
4 (i)	
4 (ii)	

5 Fig. 11b is copied below.

(i) Show that CE has length $\dfrac{(2-\sqrt{2})}{2}\,l$. **[2]**

(ii) Hence show that the square path traced out by point C (see line 67) has side length $(\sqrt{2}-1)\,l$. **[2]**

(iii) A square hole of side length 50 mm is to be cut in a sheet of plastic, using the method described in lines 69 to 71. Calculate the side length of the square hole needed in the guide plate, giving your answer correct to the nearest millimetre. **[1]**

5 (i)	
5 (ii)	
5 (iii)	

BLANK PAGE

PLEASE DO NOT WRITE ON THIS PAGE

BLANK PAGE

PLEASE DO NOT WRITE ON THIS PAGE

PLEASE DO NOT WRITE ON THIS PAGE

OCR
RECOGNISING ACHIEVEMENT

Tuesday 17 January 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4754B Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

4733160112

Duration: Up to 1 hour

INFORMATION FOR CANDIDATES

- This insert contains the text for use with the questions.
- This document consists of **8** pages. Any blank pages are indicated.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

- Do not send this insert for marking; it should be retained in the centre or recycled. Please contact OCR Copyright should you wish to re-use this document.

Curves of Constant Width

Introduction

Imagine you need to move a heavy object over level ground. You could rest your object on a horizontal platform and use cylindrical rollers, all with the same radius, as shown in Fig. 1.

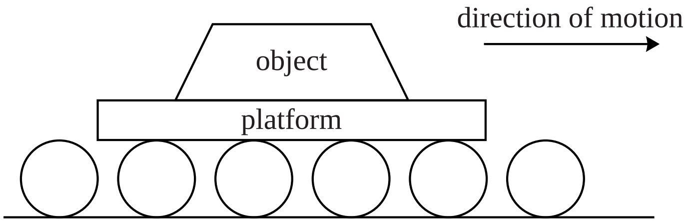


Fig. 1

Using cylindrical rollers in this way would ensure that the motion is smooth and horizontal; the object remains at the same height above the ground at all times.

If the cross-section of each roller was an ellipse, rather than a circle, then the motion would not be smooth. Fig. 2 shows different orientations of an elliptical roller.

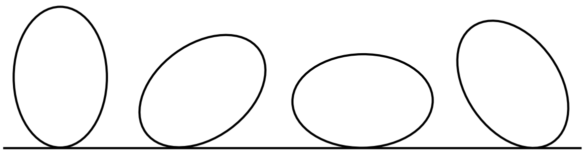


Fig. 2

You might be surprised to learn that there are infinitely many shapes which, like the circle, form the cross-section of a roller that would produce smooth horizontal motion of the object. These shapes are said to have *constant width* and the boundary of such a shape is called *a curve of constant width*. In this article you will be introduced to several types of curves of constant width.

Reuleaux triangle

The simplest non-circular curve of constant width, shown in Fig. 3, is named after Franz Reuleaux (1829–1905), a German mathematician and engineer. It is based on an equilateral triangle of side length l on which three circular arcs are drawn; each arc is centred on one vertex and passes through the other two vertices.

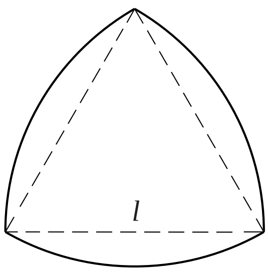


Fig. 3

If a roller with this cross-section rolled along a horizontal surface then the highest point would always be at a height l above the surface. This is illustrated in Fig. 4.

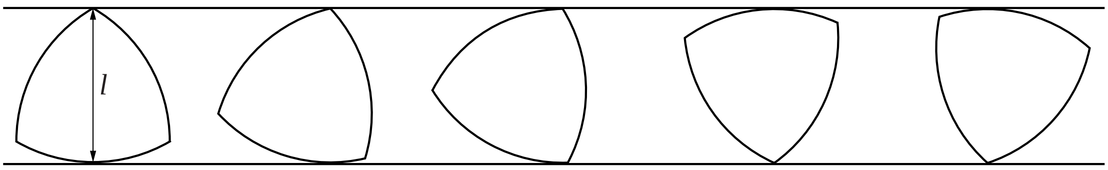


Fig. 4

In one revolution, assuming no slipping, the roller would move forward a distance πl, the same distance as for a cylindrical roller of diameter l.

In a similar way, arcs can be added to any regular polygon with an odd number of sides to make a curve of constant width. A Reuleaux pentagon, with constant width l, is shown in Fig. 5. In this curve, each arc is centred on the opposite vertex of the pentagon; for example, the arc CD has centre A and radius l.

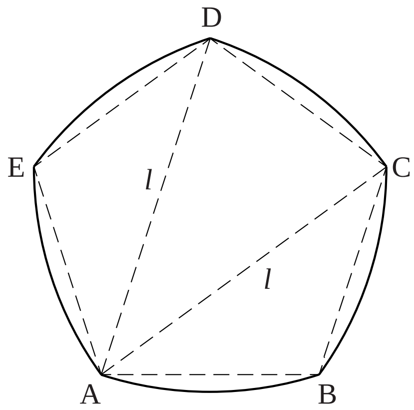


Fig. 5

Many mathematical properties of curves of constant width are known. Two of particular interest are given here.

- Every curve of constant width l has perimeter πl.
- Of all the curves of constant width l, the circle encloses the greatest area, $\frac{\pi}{4}l^2 \approx 0.785l^2$, and the Reuleaux triangle encloses the smallest area, $\left(\frac{\pi - \sqrt{3}}{2}\right) l^2 \approx 0.705l^2$.

Smooth curves of constant width

With the exception of the circle, the curves of constant width met so far have been created by constructing arcs on the sides of certain regular polygons. At each vertex of the regular polygon, two arcs meet but they do so in such a way that the curve is not smooth. For the Reuleaux pentagon in Fig. 6 there are two tangents at vertex A, one on each arc. The angle between these tangents is 144°.

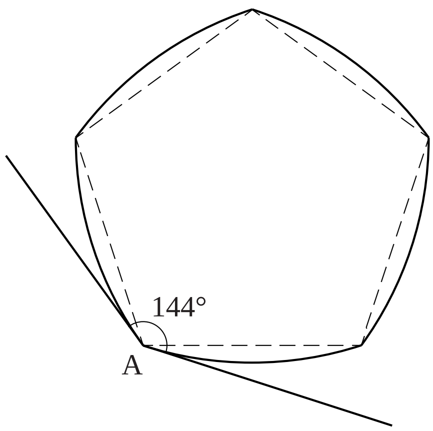


Fig. 6

It is possible to construct curves of constant width which are smooth at all points; one way of doing this is as follows.

- Draw an equilateral triangle ABC with side length R
- Extend the sides a distance r beyond each vertex, as shown in Fig. 7a
- Construct two circular arcs centred on A: arc A_1A_2 with radius r and arc B_2C_1 with radius $R + r$
- Construct similar arcs centred on B and C to give the curve shown in Fig. 7b.

This curve has constant width $R + 2r$ and is smooth. Every point on the curve has a unique tangent and the distance between parallel tangents is constant.

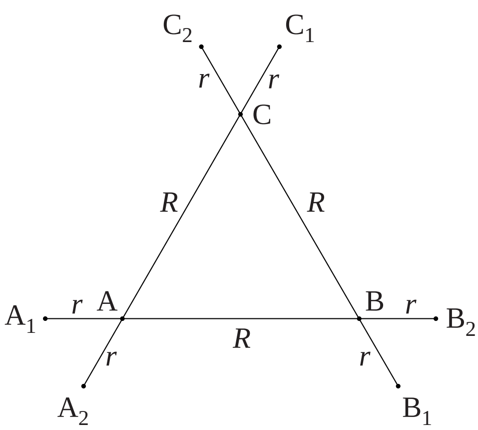


Fig. 7a

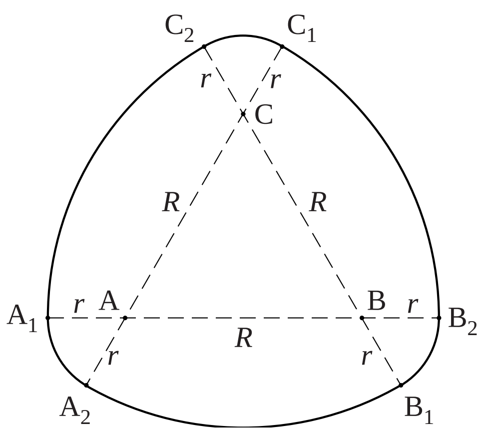


Fig. 7b

This method can be used on other regular polygons with an odd number of sides. One example of this is shown in Fig. 8. Notice that it is the longest diagonals of the heptagon that are extended.

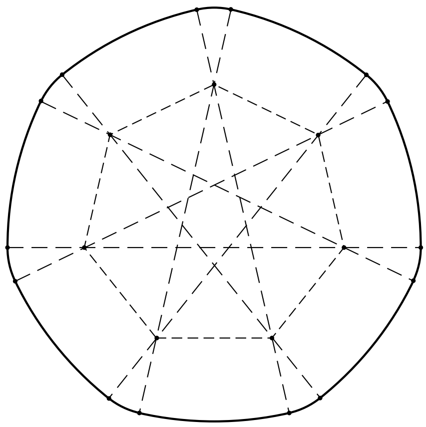


Fig. 8

Tracing out a locus

Any curve of constant width can turn inside a square; throughout the motion, the curve will always be in contact with all four sides of the square. This is illustrated for various positions of the Reuleaux triangle in Fig. 9.

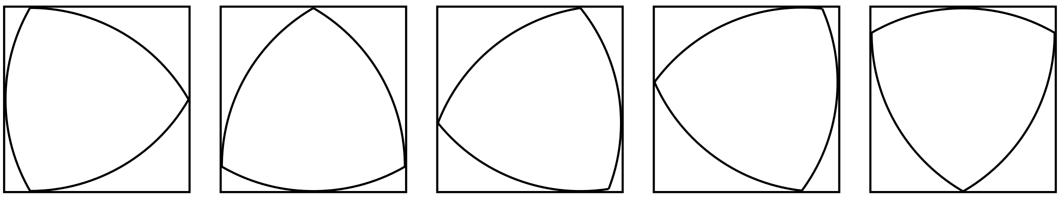


Fig. 9

If you trace the path followed by a vertex of the Reuleaux triangle, you will find that the locus is close to a square. This is shown in Fig. 10a; the locus of vertex P is made up of four straight line segments joined by rounded corners.

Fig. 10b shows the locus of the midpoint, Q, of one side of the equilateral triangle; this, too, is close to a square.

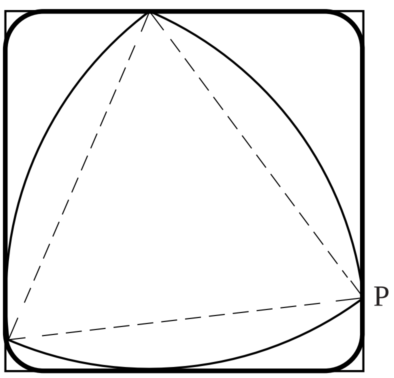


Fig. 10a

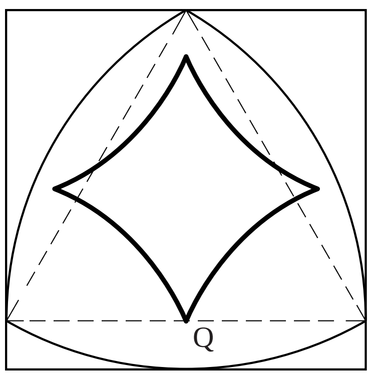


Fig. 10b

Other points in the Reuleaux triangle trace out other paths but none of these is a perfect square. This suggests the following question.

Is it possible to design a shape of constant width that contains a point which traces out a perfect square as the shape turns inside a square?

The answer to this question is ‘Yes’ and the curve is described below.

Drilling a square hole

Figs. 11a and 11b show how to construct a particular curve of constant width based on an isosceles triangle.

In triangle ABC, AB = l and angle CAB = angle CBA = 45°. Sides AC and BC are extended so that AE = BD = l. Arc BE has centre A, arc DA has centre B and arcs AB and ED have centre C.

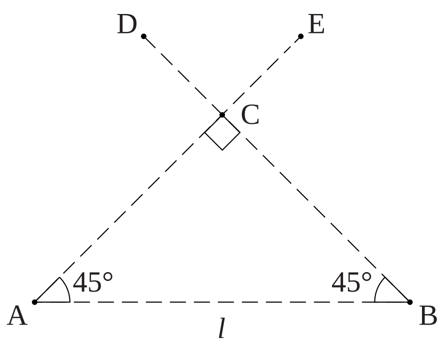


Fig. 11a

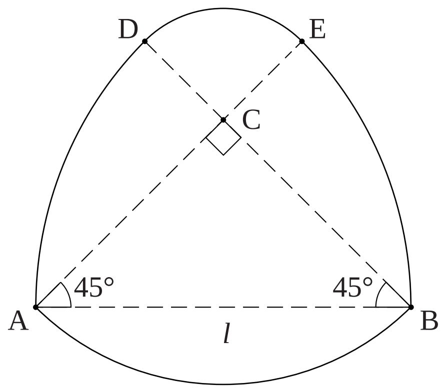


Fig. 11b

Like any curve of constant width l, the curve shown in Fig. 11b can turn inside a square of side length l. Fig. 12 shows this curve in various positions as it turns inside a square.

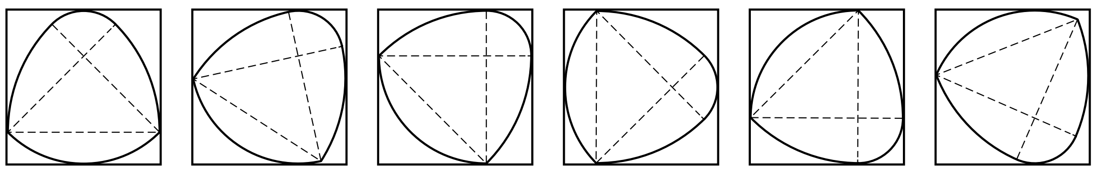


Fig. 12

The arc labelled DE in Fig. 11b remains in contact with the square throughout this motion. It follows that the locus of C is a perfect square.

This shape is used to drill a square hole.

A drill bit, with cross-section as shown in Fig. 11b, has a cutting tool at C. A metal guide plate, in which a square hole of side length l has been cut, is placed parallel to the material to be drilled. As the drill bit turns inside the guide plate, the cutting tool cuts out a square hole.

Other applications

Awareness of the existence of curves of constant width is important in engineering. Engineers rely on precision tools, many of which must be circular. To test that an object is circular, it is not sufficient simply to check that its width is constant; that would only imply that the object was one of the many shapes of constant width. Other means of checking for circularity, such as using circular templates, are needed.

It is possible that you are currently in possession of several shapes of constant width. Both the 50 pence and 20 pence coins have constant width (see Fig. 13). These coins were designed in this way so that they can easily be identified when used in machines; their width can be measured in any orientation as they move through the machine.

Fig. 13