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RECOGNISING ACHIEVEMENT

Thursday 14 June 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4754A Applications of Advanced Mathematics (C4) Paper A

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4754A
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

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INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.
- This paper will be followed by **Paper B: Comprehension**.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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Section A (36 marks)

1 Solve the equation $\frac{4x}{x+1} - \frac{3}{2x+1} = 1$. **[5]**

2 Find the first four terms in the binomial expansion of $\sqrt{1+2x}$. State the set of values of x for which the expansion is valid. **[5]**

3 The total value of the sales made by a new company in the first t years of its existence is denoted by £V. A model is proposed in which the rate of increase of V is proportional to the square root of V. The constant of proportionality is k.

(i) Express the model as a differential equation.

Verify by differentiation that $V = (\frac{1}{2}kt + c)^2$, where c is an arbitrary constant, satisfies this differential equation. **[4]**

(ii) The value of the company's sales in its first year is £10 000, and the total value of the sales in the first two years is £40 000. Find V in terms of t. **[4]**

4 Prove that $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$. **[4]**

5 Given the equation $\sin(x + 45°) = 2\cos x$, show that $\sin x + \cos x = 2\sqrt{2}\cos x$.

Hence solve, correct to 2 decimal places, the equation for $0° \leqslant x \leqslant 360°$. **[6]**

6 Solve the differential equation $\frac{dy}{dx} = \frac{y}{x(x+1)}$, given that when $x = 1$, $y = 1$. Your answer should express y explicitly in terms of x. **[8]**

Section B (36 marks)

7 Fig. 7a shows the curve with the parametric equations

$$x = 2\cos\theta, \quad y = \sin 2\theta, \quad -\tfrac{\pi}{2} \leqslant \theta \leqslant \tfrac{\pi}{2}.$$

The curve meets the x-axis at O and P. Q and R are turning points on the curve. The scales on the axes are the same.

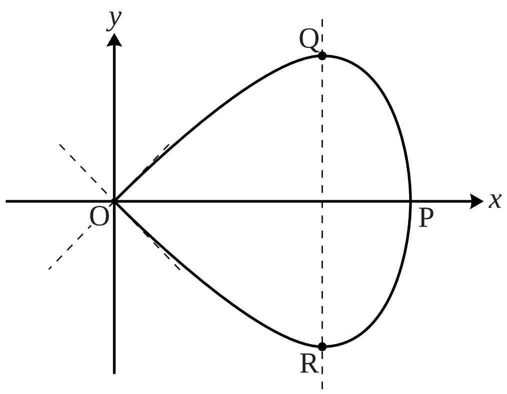

Fig. 7a

(i) State, with their coordinates, the points on the curve for which $\theta = -\frac{\pi}{2}$, $\theta = 0$ and $\theta = \frac{\pi}{2}$. **[3]**

(ii) Find $\dfrac{dy}{dx}$ in terms of θ. Hence find the gradient of the curve when $\theta = \frac{\pi}{2}$, and verify that the two tangents to the curve at the origin meet at right angles. **[5]**

(iii) Find the exact coordinates of the turning point Q. **[3]**

When the curve is rotated about the x-axis, it forms a paperweight shape, as shown in Fig. 7b.

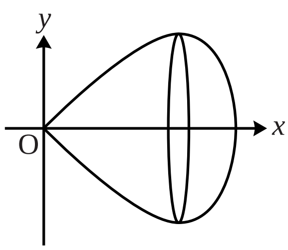

Fig. 7b

(iv) Express $\sin^2\theta$ in terms of x. Hence show that the cartesian equation of the curve is $y^2 = x^2(1 - \frac{1}{4}x^2)$. **[4]**

(v) Find the volume of the paperweight shape. **[4]**

8 With respect to cartesian coordinates Oxyz, a laser beam ABC is fired from the point A(1, 2, 4), and is reflected at point B off the plane with equation $x + 2y - 3z = 0$, as shown in Fig. 8. A′ is the point (2, 4, 1), and M is the midpoint of AA′.

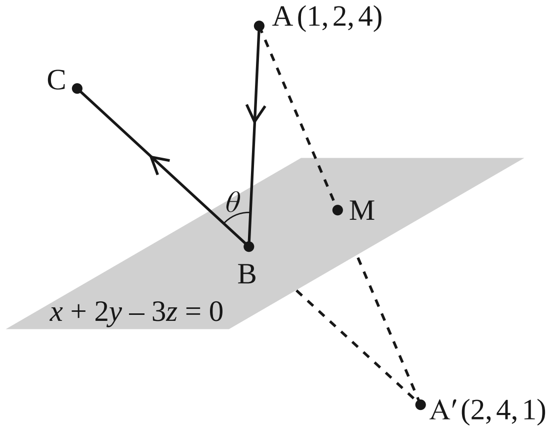

Fig. 8

(i) Show that AA′ is perpendicular to the plane $x + 2y - 3z = 0$, and that M lies in the plane. **[4]**

The vector equation of the line AB is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(ii) Find the coordinates of B, and a vector equation of the line A′B. **[6]**

(iii) Given that A′BC is a straight line, find the angle θ. **[4]**

(iv) Find the coordinates of the point where BC crosses the Oxz plane (the plane containing the x- and z-axes). **[3]**

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A2 GCE MATHEMATICS (MEI)

4754B Applications of Advanced Mathematics (C4) Paper B: Comprehension

Candidates answer on the Question Paper.

OCR supplied materials:
- Insert (inserted)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator
- Rough paper

Duration: Up to 1 hour

Candidate forename		Candidate surname	

Centre number						Candidate number				

INSTRUCTIONS TO CANDIDATES

- The Insert will be found in the centre of this document.
- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Write your answer to each question in the space provided. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- The Insert contains the text for use with the questions.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You may find it helpful to make notes and to do some calculations as you read the passage.
- You are **not** required to hand in these notes with your Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **18**.
- This document consists of **8** pages. Any blank pages are indicated.

1 Use Fig. 4 to estimate the number of 50–54 year olds in the UK in 2001. (These were born in the post World War 2 baby boom.) **[1]**

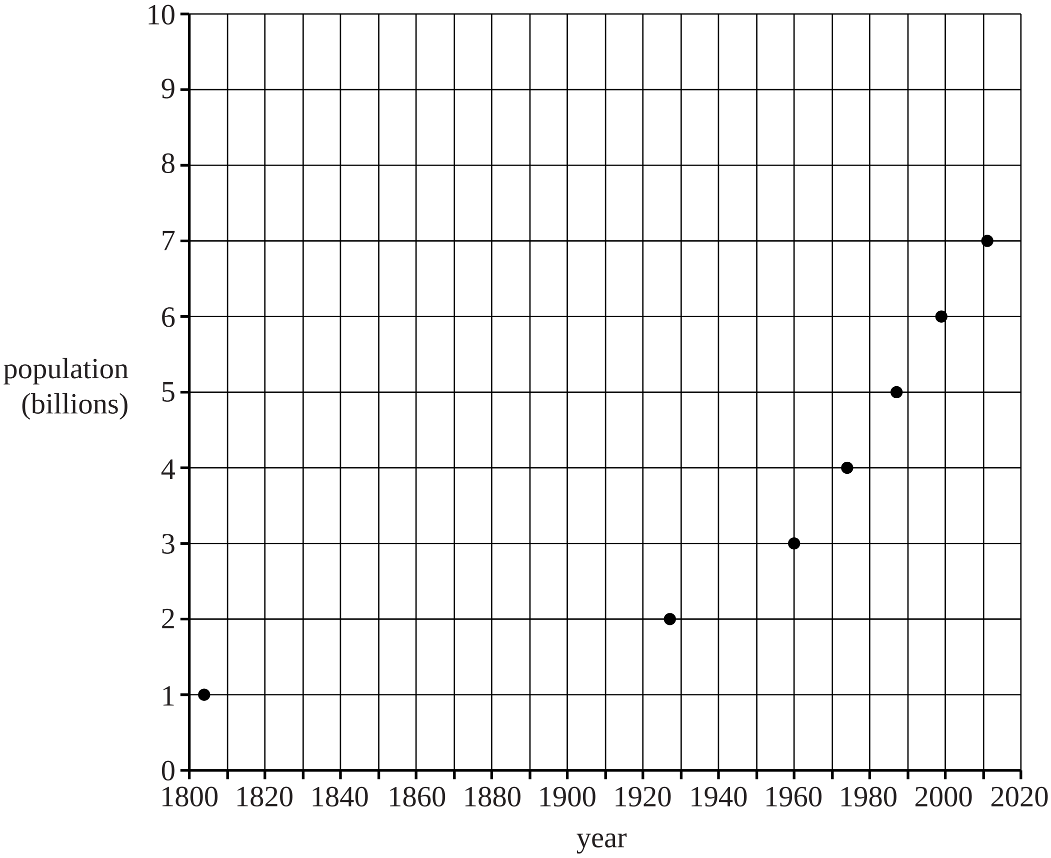
1	

2 A copy of Fig. 2 is given below.

(i) Join the points with a curve and hence estimate the rate of population growth in the year 1927 in people per year. **[3]**

(ii) Estimate this rate as a percentage of the population at that time. **[2]**

2 (i)



2 (ii)

3 **(i)** In line 21, the solution of the differential equation $\frac{\mathrm{d}p}{\mathrm{d}t} = kp$ is stated to be $p = p_0 \mathrm{e}^{kt}$.

Use integration to derive this result. **[3]**

(ii) The article then goes on to say

"If a model is to be valuable in this context, it must be possible to use it to predict the size of the world population in the future. So, as a test case, the first two data points in Table 1 should allow the later values to be predicted. These data points are

$$1804 \qquad t = 0,\ p = p_0 = 10^9,$$
$$1927 \qquad t = 123,\ p = 2\times10^9,$$

and these correspond to $k = 0.005\,63\ldots$."

Show how this value of k is obtained. **[2]**

3 (i)	
3 (ii)	

4 In Table 6, the population profile of an imaginary country was predicted. Complete the table subject to the same general assumptions except that, after 2010:

- the average number of children per female is 2.2;
- 60% of those in the 40–59 age group survive into the 60–79 age group;
- 20% of those in the 60–79 age group survive into the 80+ age group. **[3]**

4

Age group	2010	2030	2050	2070
80+	1			
60–79	10			
40–59	20			
20–39	20			
0–19	20			
Total	**71**			

As in Table 6, the figures are in millions.

5 In constructing Table 6, some assumptions were made about the proportion of people surviving from one age group to the next. Use Table 6 to find

(i) the proportion of people in the 40–59 age group surviving into the 60–79 age group, **[1]**

(ii) the proportion of those in the 60–79 age group surviving into the 80+ age group. **[1]**

5 (i)

5 (ii)

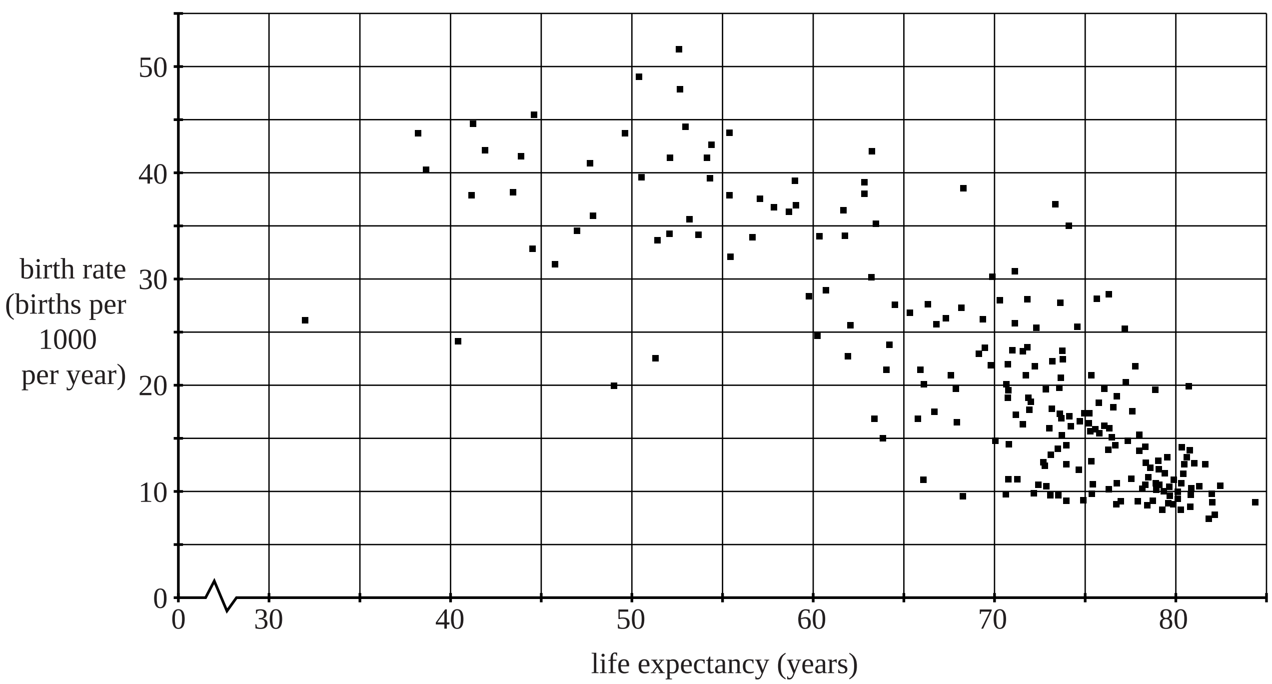
6 This table refers to the UK. It gives life expectancy and birth rate every 20 years from 1901 to 2001.

Year	**Life expectancy**	**Birth rate (births / 1000)**
1901	47	28.5
1921	58	22.7
1941	64	14.5
1961	71	17.8
1981	74	12.9
2001	78	12.0

Explain how these data relate to the conclusions of the article. **[2]**

[A copy of Fig. 7 is given below. You do not need to use it but may find it helpful.]

6	



Thursday 14 June 2012 – Morning

A2 GCE MATHEMATICS (MEI)

4754B Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

INFORMATION FOR CANDIDATES

- This Insert contains the text for use with the questions.
- This document consists of **8** pages. Any blank pages are indicated.

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The world population

Population pressure on our planet

During the last 200 years, the human population has increased by a factor of about 7. Table 1 gives the years when it reached 1, 2, 3 and so on billions of people, where 1 billion is 10^9.

Year	1804	1927	1960	1974	1987	1999	2011
Population (billions)	1	2	3	4	5	6	7

Table 1 World population

The increase in population is placing rising demands on the resources of our planet and on the whole eco-system that supports us. This raises very important questions.

- Is the world's population going to continue to increase indefinitely or will there be a limit?
- Will the world's population reach a level that the planet is unable to support?
- Should we be taking measures to restrict the world's population, and if so what?

The first two of these questions require mathematical modelling of the situation. The third involves political and ethical decisions which should be informed by that modelling.

The modelling involved is complicated; this article introduces some of the issues involved.

The exponential model

A simple mathematical model is that the world's population is increasing at a rate which is directly proportional to its existing size,

$$\frac{dp}{dt} = kp \, ,$$

where p is the number of people,
t is time, measured in years,
k is a constant.

The solution of this differential equation is

$$p = p_0 e^{kt}$$

where p_0 is the population at the time from which t is measured.

If a model is to be valuable in this context, it must be possible to use it to predict the size of the world population in the future. So, as a test case, the first two data points in Table 1 should allow the later values to be predicted. These data points are

$$1804 \qquad t = 0,\ p = p_0 = 10^9,$$
$$1927 \qquad t = 123,\ p = 2\times10^9,$$

and these correspond to $k = 0.005\,63\ldots$.

With this value of k, this model would predict that the population in 2011 would be 3.2 billion but in fact it was 7 billion. This model, based on the first two data points, is clearly not suitable.

In fact no exponential model fits the data in Table 1 well. You can see this just by looking at the graph of the data in Fig. 2. The graph of an exponential function is a curve which gets steeper and steeper but for the last 50 years this graph is virtually a straight line, indicating a constant rate of growth.

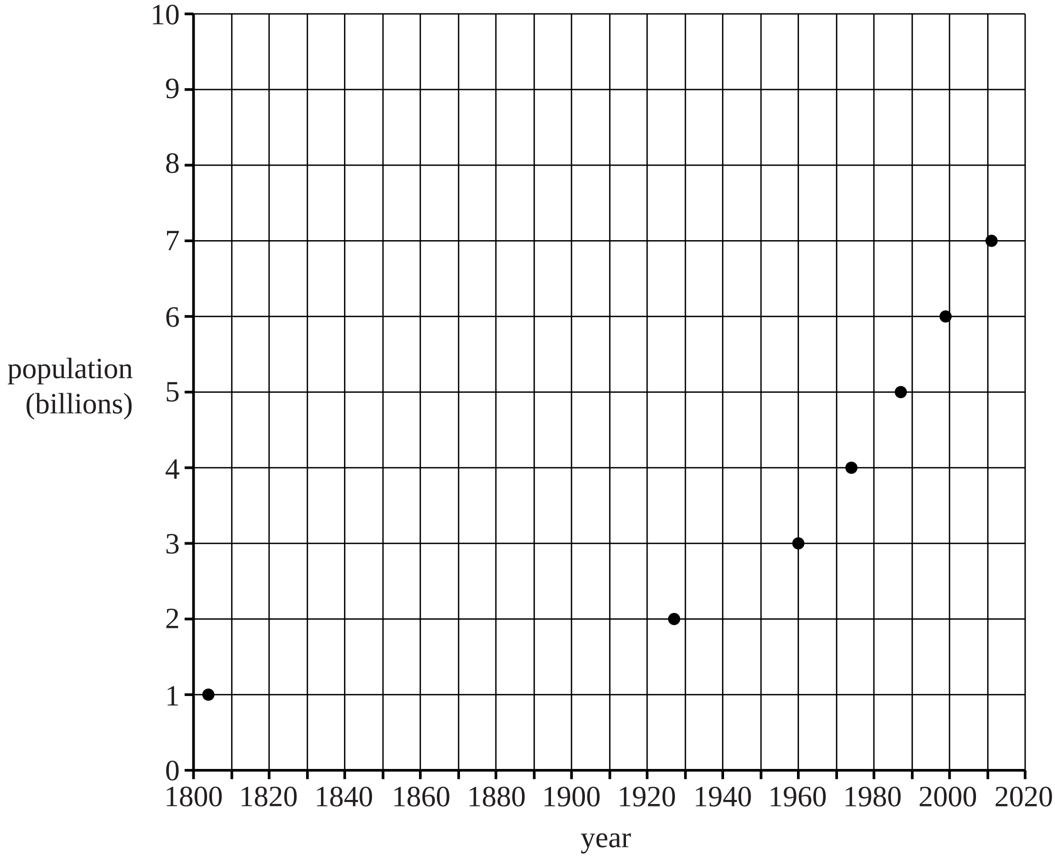

Fig. 2 World population from 1800 to the present

The logistic model

A standard mathematical model for a population which increases towards a limiting value of m is given by the differential equation

$$\frac{dp}{dt} = kp(m - p) .$$

This is known as the *logistic equation*. A typical solution curve is shown in Fig. 3.

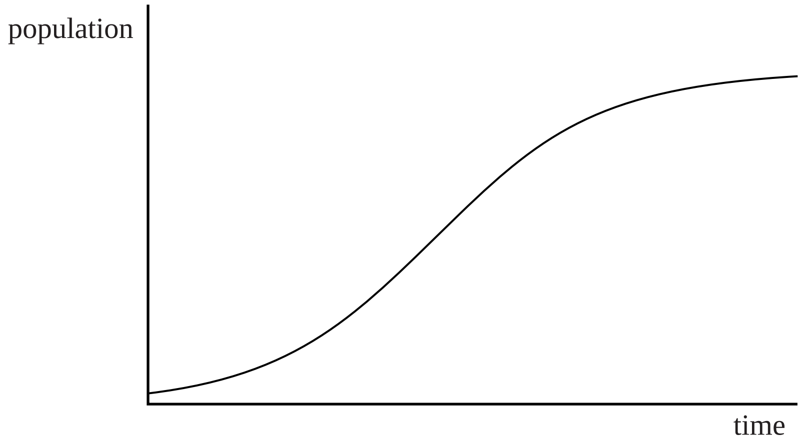

Fig. 3 The logistic model

While this looks as though it starts with the same sort of shape as a curve through the data points in Fig. 2, the resemblance is only superficial; it is not actually possible to find values of k and m that produce anything like a good fit. So this model is also unsatisfactory.

Like the previous model, this is an attempt to find a simple, neat solution to a very complicated problem. A different approach is needed and a starting point is provided by population profiles.

Population profiles

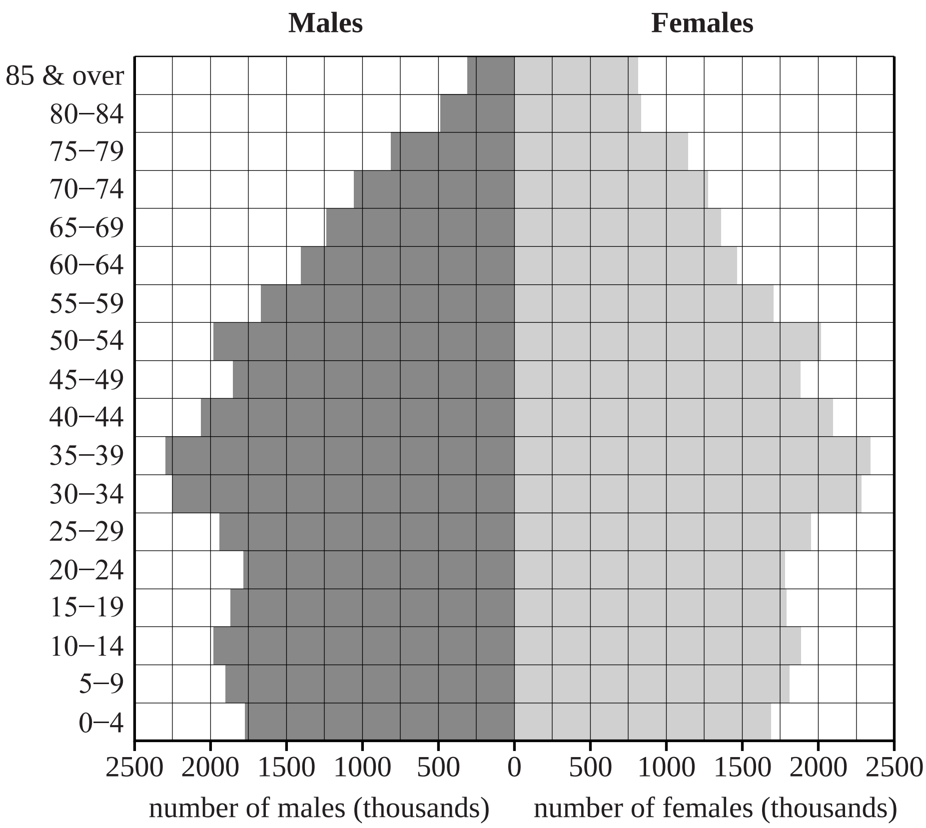

Fig. 4 Population profile of the UK in 2001

Population profiles are often illustrated by population pyramids, like that in Fig. 4. The lengths of the horizontal bars indicate the numbers of males and females in the UK population in 2001, in 5-year age intervals. In this case, the numbers on the horizontal scale are in thousands. Those on the vertical scale refer to age in completed years so that, for example, 10–14 means from 10 years 0 days to 14 years 364 days.

The UK population profile shows that in 2001 the number of children in the 0– 4 age range was among the lowest for 50 years. Because there are fewer people in that age group, they in turn can be expected to have fewer children.

The shapes of the population profiles vary considerably between countries. In some countries the profiles have very wide bases, indicating large numbers of children.

It is worth noting that population figures for the UK are often affected by emigration and immigration.

Modelling using population profiles

If the individual population profiles of the large number of countries in the world are combined, a profile for the whole world emerges. It is possible to predict the changes in any country's profile in the years ahead, and hence the changes in the world's population. Each country is different and so needs to be looked at separately before combining the profiles.

The following model, for an imaginary country, is designed to highlight the key factors. Table 5 illustrates its profile in 2010 and part of that for 2030.

Age group	2010	2030
80+	1	?
60–79	10	?
40–59	20	20
20–39	20	20
0–19	20	?
Total	**71**	**?**

Table 5 Population profile of an imaginary country (in millions)

In 2010, this country has a stable population with the same numbers in the youngest three age groups, up to the age of 60; however, life expectancy is quite low with very few people reaching the age of 80.

Two of the figures for 2030 have also been filled in. The 20 million people in the 0–19 age group in 2010 will move into the 20–39 group. Similarly those in the 20–39 group will move into the 40–59 group. (It is assumed, for simplicity, that no one in these age groups dies.) What will the other figures for 2030 be?

Two different factors are involved: the birth rate and the life expectancy.

The 2010 profile in Table 5 was constructed using a number of assumptions:

- that those in the 0–19 age group are all children of females in the 20–39 group;
- that 50% of those in the 20–39 age group are female;
- that on average each female has 2 children;
- that there is no immigration or emigration.

While these assumptions are obviously somewhat artificial, particularly with regard to the age at which women have children, they are good enough to demonstrate the key features of a country's population.

Throughout the world, life expectancy is rising. The proportion of the population in Table 5 who reach the age of 80 could be expected to increase.

In Table 6, the population profile of the country in Table 5 is predicted for the next 100 years, on the basis of the following new assumptions about the birth rate and life expectancy.

- Every 20 years, each group of people moves up a level.
- The average figure of 2 children per female is assumed to fall to 1.8, from 2010 onwards.
- The proportion of those in the 40–59 age group surviving into the 60–79 group increases from the 2010 figure of 50%; similarly there is an increase in survival from the 60–79 group into the 80+ group.

The figures used in these assumptions have been chosen to illustrate the modelling process. Their use does not mean that they will actually apply to the population of any real country.

Age group	**2010**	**2030**	**2050**	**2070**	**2090**	**2110**
80+	1	4	6.4	6.4	6.4	5.76
60–79	10	16	16	16	14.4	12.96
40–59	20	20	20	18	16.2	14.58
20–39	20	20	18	16.2	14.58	13.12
0–19	20	18	16.2	14.58	13.12	11.81
Total	**71**	**78**	**76.6**	**71.18**	**64.7**	**58.23**

Table 6 Population profile of an imaginary country (in millions, to 4 significant figures)

The figures in Table 6 show the total population rising quite sharply to a maximum and then reducing, initially rather slowly but then more quickly. Patterns like this are observed in many countries. In some, like Japan, the population has passed its maximum and is declining. In most, however, it is still increasing and consequently the population of the world as a whole is still increasing.

An important feature of Table 6 is that it is based on a low birth rate of 1.8 children per female. In many countries the birth rate is much higher than the stable level of 2 children per female.

Birth rate and life expectancy

Two key factors that determine the change in a country's population have been identified as its birth rate and its life expectancy. Data show that these are closely associated. Countries with high birth rates tend to have low life expectancy and those with low birth rates have high life expectancy. This is illustrated in Fig. 7 for all 221 countries; the data were drawn from the CIA World Factbook for 2009.

Notice that in Fig. 7 the birth rate is the number of births per 1000 of the population per year. It is thus a different measure from that used so far in this article which is mean births per female over her lifetime. So, for example, a country with a population of 80 million people and 1.2 million births per year has a birth rate of 15 births per 1000 per year.

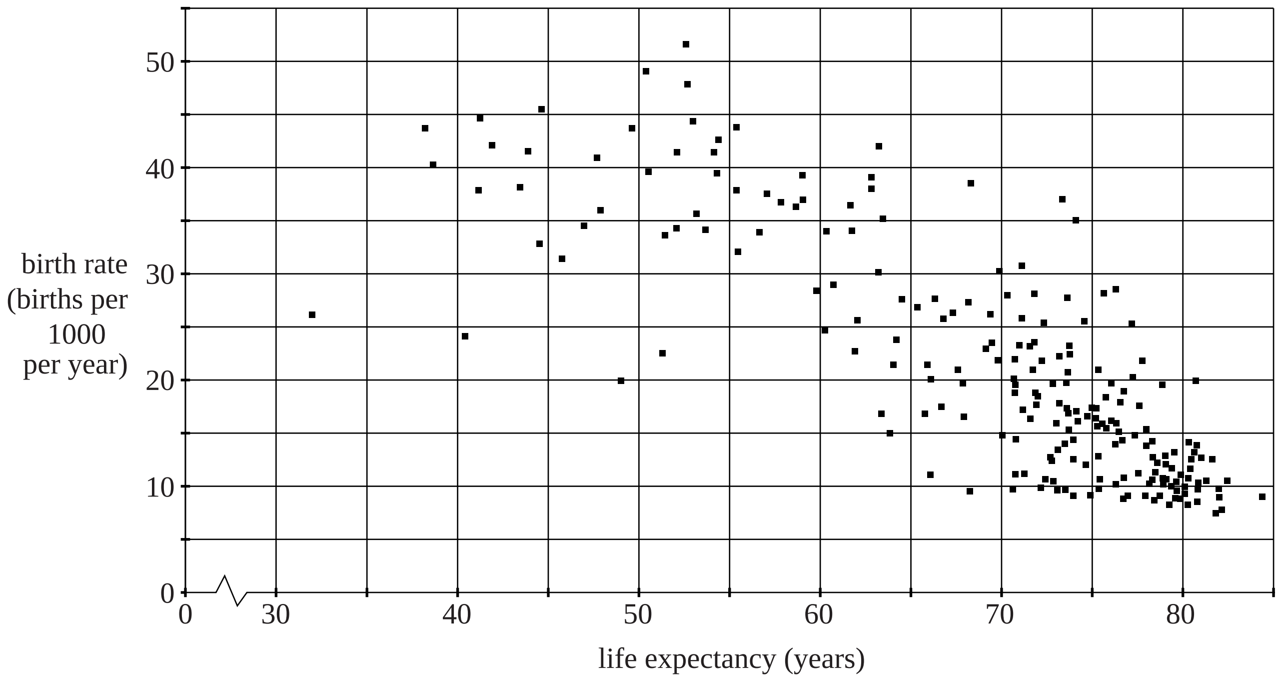

Fig. 7 Scatter diagram showing birth rate against life expectancy for the countries of the world

Conclusion

Table 8 gives the data for some selected countries in 2009.

Country	Life expectancy	Birth rate
Japan	82.12	7.64
Sweden	80.86	10.13
Italy	80.20	8.18
UK	79.01	10.65
USA	78.11	13.82
Tunisia	75.78	15.42
Jamaica	73.53	19.68
China	73.47	14.00
Brazil	71.99	18.43
India	69.89	21.76
Bangladesh	60.25	24.68
Ghana	59.85	28.58
Uganda	52.72	47.84
Afghanistan	44.64	45.46

Table 8 Life expectancy and birth rates of selected countries (2009)

The data in Table 8 illustrate the observation that countries with low birth rate and high life expectancy tend to be those with developed economies. Studies over time indicate that as they develop, countries follow a path from high birth rate and low life expectancy to low birth rate and high life expectancy. So it is reasonable to expect that at some time in the future, the world's population will attain a maximum value and then start to decline.

When that maximum occurs, and how great the population then is, will depend on how quickly countries progress along that path. Consequently modelling the world's population requires an understanding of the factors involved. Then it will be possible to determine what can be done to match the population to the planet's resources.