Oxford Cambridge and RSA

Tuesday 21 June 2022 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Time allowed: 2 hours

9122353923

You must have:
- the Printed Answer Booklet
- the Insert
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae A Level Mathematics B (MEI) (H640)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^n\mathrm{C}_1 a^{n-1}b + {}^n\mathrm{C}_2 a^{n-2}b^2 + \ldots + {}^n\mathrm{C}_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^n\mathrm{C}_r = {}_n\mathrm{C}_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1,\ n \in \mathbb{R})$

Differentiation

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x \cot x$

Quotient Rule $y = \frac{u}{v}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$

Differentiation from first principles

$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$

Integration

$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$

$\int \mathrm{f}'(x)(\mathrm{f}(x))^n\,\mathrm{d}x = \frac{1}{n+1}(\mathrm{f}(x))^{n+1} + c$

Integration by parts $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \text{si}\ \ A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Sample variance

$s^2 = \dfrac{1}{n-1}S_{xx}$ where $S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n} = \Sigma x_i^2 - n\bar{x}^2$

Standard deviation, $s = \sqrt{\text{variance}}$

The binomial distribution

If $X \sim \mathrm{B}(n, p)$ then $\mathrm{P}(X = r) = {}^n\mathrm{C}_r\, p^r q^{n-r}$ where $q = 1 - p$

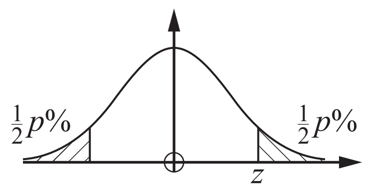
Mean of X is np

Hypothesis testing for the mean of a Normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the Normal distribution

p	10	5	2	1
z	1.645	1.960	2.326	2.576



Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

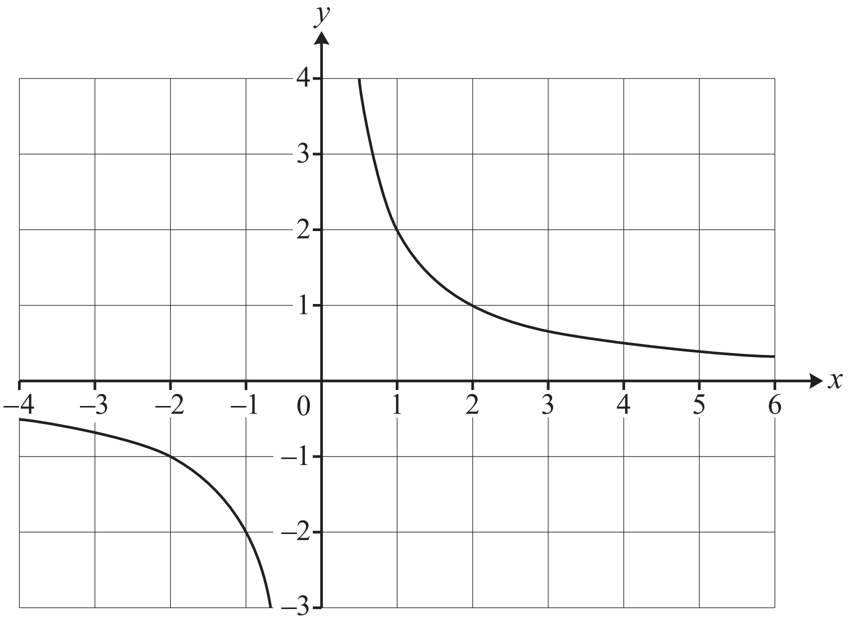
$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Answer **all** the questions.

Section A (60 marks)

1 A curve for which y is inversely proportional to x is shown below.



Find the equation of the curve. **[2]**

2 The function $f(x) = \sqrt{x}$ is defined on the domain $x \geqslant 0$.

The function $g(x) = 25 - x^2$ is defined on the domain $\mathbb{R}$.

(a) Write down an expression for $fg(x)$. **[1]**

(b) **(i)** Find the domain of $fg(x)$. **[3]**

(ii) Find the range of $fg(x)$. **[2]**

3 An infinite sequence $a_1, a_2\ a_3, \ldots$ is defined by $a_n = \frac{n}{n+1}$, for all positive integers n.

(a) Find the limit of the sequence. **[1]**

(b) Prove that this is an increasing sequence. **[3]**

4 **In this question you must show detailed reasoning.**

Determine the exact solutions of the equation $2\cos^2 x = 3\,\mathrm{si}\;\; x$ for $0 \leqslant x \leqslant 2\pi$. **[5]**

5 A curve is defined implicitly by the equation $2x^2 + 3xy + y^2 + 2 = 0$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{4x+3y}{3x+2y}$. **[3]**

(b) **In this question you must show detailed reasoning.**

Find the coordinates of the stationary points of the curve. **[4]**

6 A hot drink is cooling. The temperature of the drink at time t minutes is T°C.

The rate of decrease in temperature of the drink is proportional to $(T-20)$.

(a) Write down a differential equation to describe the temperature of the drink as a function of time. **[2]**

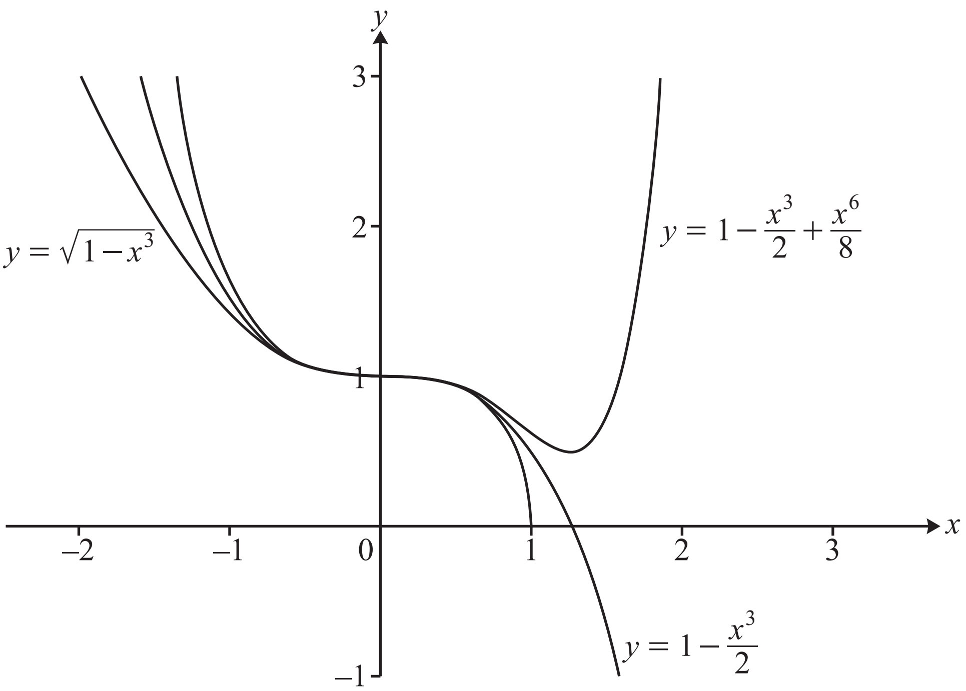
(b) When $t = 0$, the temperature of the drink is 90 °C and the temperature is decreasing at a rate of 4.9 °C per minute.

Determine how long it takes for the drink to cool from 90 °C to 40 °C. **[6]**

7 A student is trying to find the binomial expansion of $\sqrt{1-x^3}$.

She gets the first three terms as $1-\frac{x^3}{2}+\frac{x^6}{8}$.

She draws the graphs of the curves $y=\sqrt{1-x^3}$, $y=1-\frac{x^3}{2}$ and $y=1-\frac{x^3}{2}+\frac{x^6}{8}$ using software.



(a) Explain why $1-\frac{x^3}{2}+\frac{x^6}{8} \geqslant 1-\frac{x^3}{2}$ for all values of x. **[1]**

(b) Explain why the graphs suggest that the student has made a mistake in the binomial expansion. **[1]**

(c) Find the first **four** terms in the binomial expansion of $\sqrt{1-x^3}$. **[3]**

(d) State the set of values of x for which the binomial expansion in part **(c)** is valid. **[1]**

(e) Sketch the curve $y=2.5\sqrt{1-x^3}$ on the grid in the Printed Answer Booklet. **[2]**

(f) **In this question you must show detailed reasoning.**

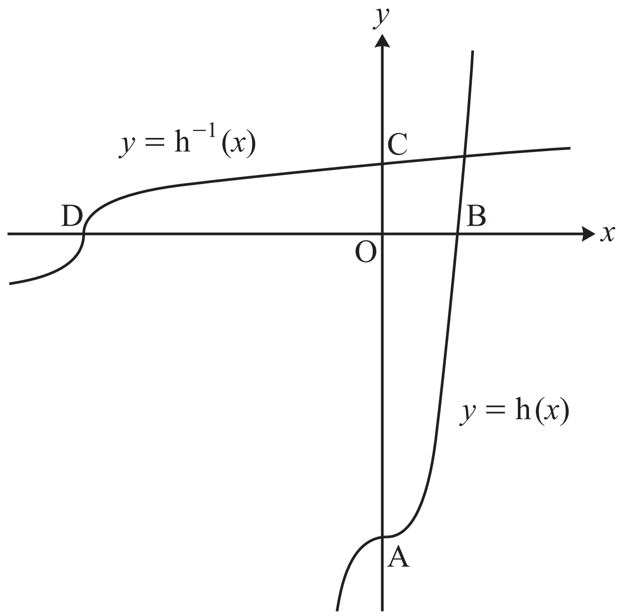
The end of a bus shelter is modelled by the area between the curve $y=2.5\sqrt{1-x^3}$, the lines $x=-0.75$, $x=0.75$ and the x-axis. Lengths are in metres.

Calculate, using your answer to part **(c)**, an approximation for the area of the end of the bus shelter as given by this model. **[4]**

8 The curves $y = \mathrm{h}(x)$ and $y = \mathrm{h}^{-1}(x)$, where $\mathrm{h}(x) = x^3 - 8$, are shown below.

The curve $y = \mathrm{h}(x)$ crosses the x-axis at B and the y-axis at A.

The curve $y = \mathrm{h}^{-1}(x)$ crosses the x-axis at D and the y-axis at C.



(a) Find an expression for $\mathrm{h}^{-1}(x)$. **[2]**

(b) Determine the coordinates of A, B, C and D. **[5]**

(c) Determine the equation of the perpendicular bisector of AB. Give your answer in the form $y = mx + c$, where m and c are constants to be determined. **[4]**

(d) Points A, B, C and D lie on a circle.

Determine the equation of the circle. Give your answer in the form $(x - a)^2 + (y - b)^2 = r^2$, where a, b and r^2 are constants to be determined. **[5]**

Answer **all** the questions.

Section B (15 marks)

The questions in this section refer to the article on the Insert. You should read the article before attempting the questions.

9 Show that $y = x$ has the same gradient as $y = \sin x$ when $x = 0$, as stated in line 5. **[2]**

10 In this question you must show detailed reasoning.

Fig. C2.2 indicates that the curve $y = \dfrac{4x(\pi - x)}{\pi^2} - \sin x$ has a stationary point near $x = 3$.

- Verify that the x-coordinate of this stationary point is between 2.6 and 2.7.
- Show that this stationary point is a maximum turning point. **[5]**

11 Show that, for the angle 45°, the formula $\sin\theta \approx \dfrac{4\theta(180-\theta)}{40500-\theta(180-\theta)}$ given in line 28 gives the same approximation for the sine of the angle as the formula $\sin x \approx \dfrac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ given in line 23. **[3]**

12 (a) Show that $\cos x = \sin\left(x + \dfrac{\pi}{2}\right)$. **[2]**

(b) Hence show that $\sin x \approx \dfrac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ gives the approximation $\cos x \approx \dfrac{\pi^2 - 4x^2}{\pi^2 + x^2}$, as stated in line 31. **[3]**

END OF QUESTION PAPER

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Tuesday 21 June 2022 – Afternoon

A Level Mathematics B (MEI)

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Insert

Time allowed: 2 hours

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INSTRUCTIONS

- Do **not** send this Insert for marking. Keep it in the centre or recycle it.

INFORMATION

- This Insert contains the article for Section B.
- This document has **4** pages.

Approximating the sine function

Small angles

For a small angle x radians, the approximation $\sin x \approx x$ is valid. The curve $y = \sin x$ and the straight line $y = x$ are shown in **Fig. C1.1**. **Fig. C1.2** shows the curve $y = x - \sin x$. Inspection of the graphs suggests that x is a reasonable approximation for $\sin x$ for $-0.5 \leqslant x \leqslant 0.5$ and also that $y = x$ has the same gradient as $y = \sin x$ when $x = 0$.

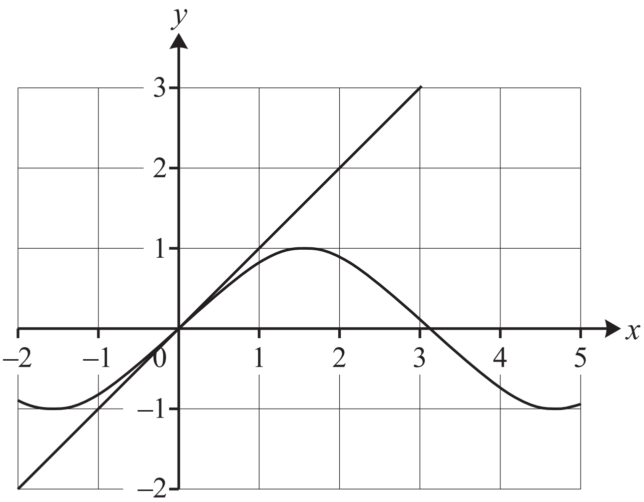

Fig. C1.1

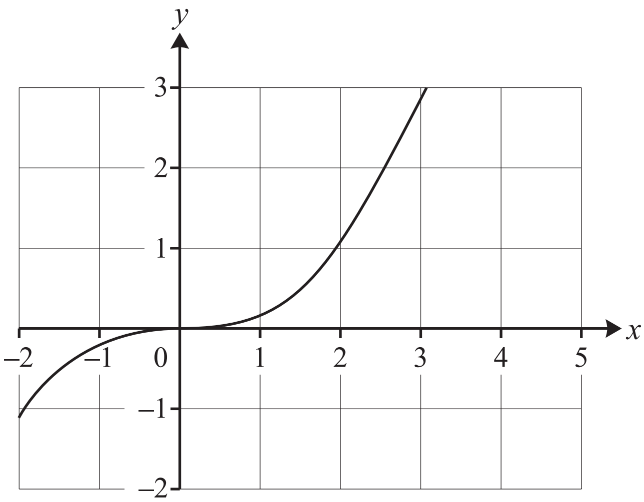

Fig. C1.2

Calculating sin *x*

Trigonometric functions, including $\sin x$, are widely used so it is useful to be able to calculate the value of the sine of any angle accurately and quickly. This is easily done nowadays using a calculator but this was not possible in the past. The linear function, $y = x$, is only a reasonable approximation for $y = \sin x$ for values of x close to zero. Perhaps using a higher degree polynomial would give a reasonable approximation for a wider range of values of x.

Fig. C2.1 shows the curve $y = \sin x$ and the quadratic curve which goes through the points (0, 0), $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$. The equation of this curve is $y = \frac{4x(\pi - x)}{\pi^2}$. **Fig. C2.2** shows the curve $y = \frac{4x(\pi - x)}{\pi^2} - \sin x$.

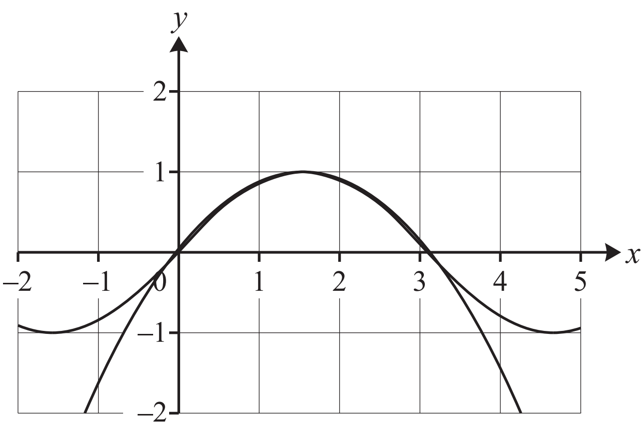

Fig. C2.1

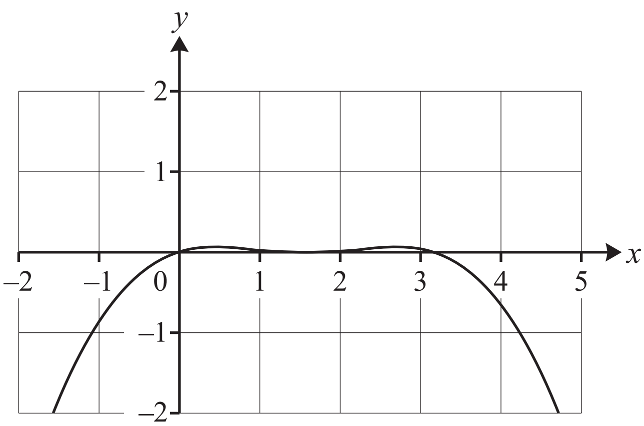

Fig. C2.2

The quadratic function seems to be a reasonably good approximation for $\sin x$ in the interval $0 \leqslant x \leqslant \pi$. However, calculating percentage errors for selected values of x shows that the percentage errors made by using the quadratic function as an approximation to $\sin x$ are quite high for values of x close to zero or π.

The spreadsheet in **Fig. C3** shows values of x in column A, with the corresponding values of $\sin x$ and the quadratic function $\frac{4x(\pi - x)}{\pi^2}$ in columns B and C. Columns D and E show the percentage errors in using x and the quadratic as approximations for $\sin x$.

	A	B	C	D	E
1	x	sin(x)	quadratic	% error for x	% error for quadratic
2	0	0	0		
3	0.1	0.099833	0.123271	0.166861	23.476799
4	0.2	0.198669	0.238437	0.669791	20.016773
5	0.3	0.295520	0.345496	1.515901	16.911206
6	0.4	0.389418	0.444450	2.717298	14.131825

Fig. C3

A better approximation

The approximation $\sin x \approx \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ was discovered by an Indian mathematician named Bhaskara in the 7th century. It is not known how Bhaskara derived the formula but it can be seen that the curve $y = \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ is symmetrical about $x = \frac{\pi}{2}$ and goes through the points (0, 0), $\left(\frac{\pi}{2}, 1\right)$ and $(\pi, 0)$. **Fig. C4** shows the curves $y = \sin x$ and $y = \frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$. Radians were not in use until the 18th century; Bhaskara gave the formula for an angle θ degrees as $\sin\theta \approx \frac{4\theta(180 - \theta)}{40500 - \theta(180 - \theta)}$.

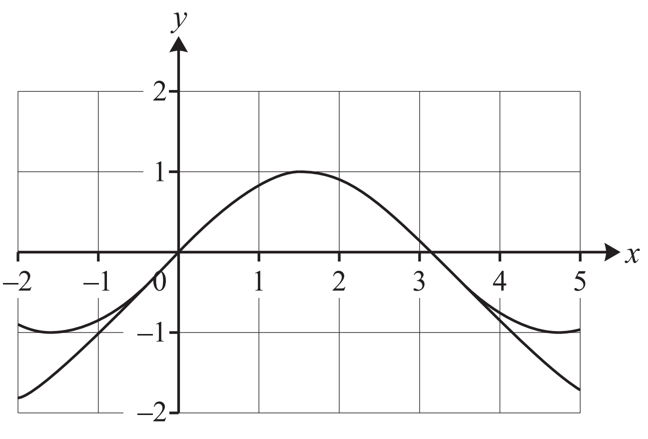

Fig. C4

The percentage error in approximating $\sin x$ by $\frac{16x(\pi - x)}{5\pi^2 - 4x(\pi - x)}$ is less than 2% throughout the interval $0 \leqslant x \leqslant \pi$. The Bhaskara approximation for $\sin x$ can be used to derive the following approximation for $\cos x$; $\cos x \approx \frac{\pi^2 - 4x^2}{\pi^2 + x^2}$.

