GCE

Further Mathematics A

Y541/01: Pure Core 2

Advanced GCE

Mark Scheme for Autumn 2021

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

Annotations and abbreviations

Annotation in RM assessor	**Meaning**
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
BP	Blank Page
NBOD	Benefit of doubt not given
Highlighting	
Other abbreviations in mark scheme	**Meaning**
dep*	Mark dependent on a previous mark, indicated by *. The * may be omitted if only one previous M mark
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working
AG	Answer given
awrt	Anything which rounds to
BC	By Calculator
DR	This question included the instruction: In this question you must show detailed reasoning.

Question			Answer	Marks	AO	Guidance	
1			$\begin{pmatrix} -12 & -1 & 5 \\ -1 & -20 & 3 \\ 7-6a & 3a-2 & -4a-5 \end{pmatrix}$ or $\begin{pmatrix} -4a & -1 & 5 \\ 11-4a & -20 & 3 \\ -8-a & 7 & -17 \end{pmatrix}$ seen	M1	1.1	Either product **AB** or **BA** calculated (but not if assigned incorrectly). Alternatively: equivalent correct useful entries calculated for both	Condone 3 errors or omissions This mark can be implied by sight of a correct equation
			$-12 = -4a$ or $-1 = 11 - 4a$ or $7 - 6a = -8 - a$ or $3a - 2 = 7$ or $-4a - 5 = -17$	M1	1.1	Finding matrix products both ways and equating entries usefully	This mark can be implied by sight of a correct equation even if other entries or equations are wrong.
			$a = 3$	A1 **[3]**	2.2a		Cannot be awarded if either **AB** or **BA** has more than 3 errors

Question			Answer	Marks	AO	Guidance	
2	(a)	(i)	DR $3z_1 + 4z_2 = 3(3-7\text{i}) + 4(2+4\text{i}) = 17 - 5\text{i}$	B1 [1]	1.1		
		(ii)	DR $z_1z_2 = (3-7\text{i})(2+4\text{i}) = 6 + 12\text{i} - 14\text{i} - 28(-1)$	M1	1.1	Attempted expansion with $\text{i}^2 = -1$ used and at least 3 correctly expanded terms	$-28(-1)$ can be simply $+28$
			$= 34 - 2\text{i}$	A1 [2]	1.1		
		(iii)	DR $\dfrac{z_1}{z_2} = \dfrac{3-7\text{i}}{2+4\text{i}} = \dfrac{3-7\text{i}}{2+4\text{i}} \times \dfrac{2-4\text{i}}{2-4\text{i}}$	M1	1.1	Multiplying top and bottom by (real multiple of) conjugate of bottom	
			$= \dfrac{6-12\text{i}-14\text{i}-28}{4+16} = \dfrac{-22-26\text{i}}{20} = -\dfrac{11}{10} - \dfrac{13}{10}\text{i}$	A1 [2]	1.1	Must see some evidence of expansion	Allow $\dfrac{-11-13\text{i}}{10}$ or $-\dfrac{11+13\text{i}}{10}$
	(b)		DR $\sqrt{3^2 + (-7)^2}$ or $\tan^{-1}\left(\dfrac{-7}{3}\right)$	M1	1.1	Explicit working must be seen	Other trig calculations could be sufficient for M1 provided that these are being used to find the argument.
			$\lvert z_1 \rvert = \sqrt{58}$ or awrt 7.62 or $\arg z_1 =$ awrt -1.17 or 5.12 rads	A1	1.1		
			$z_1 = \sqrt{58}\text{cis}(-1.17)$ or $z_1 = \sqrt{58}\text{e}^{-1.17\text{i}}$ or $z_1 = \sqrt{58}(\cos(-1.17) + \text{i}\sin(-1.17))$ or $\left[\sqrt{58}, -1.17\right]$	A1 [3]	2.5	Must be in correct form with $\sqrt{58}$ exact and could be awrt 5.12 instead of -1.17.	Do not condone degrees Condone round brackets

Question			Answer	Marks	AO	Guidance	
3	(a)		$\left(\begin{pmatrix}1\\-3\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\-2\end{pmatrix}\right).\begin{pmatrix}2\\-5\\-3\end{pmatrix}=4$	M1	1.1	Substituting the expression for a point on the line into the equation of the plane	
			$2+15-9+\lambda(6-10+6)=4$ $8+2\lambda=4 \Rightarrow 2\lambda=-4 \Rightarrow \lambda=-2$ so	M1	1.1	Dotting out to form and solve equation in λ	
			$\mathbf{r}=\begin{pmatrix}1\\-3\\3\end{pmatrix}+-2\begin{pmatrix}3\\2\\-2\end{pmatrix}=\begin{pmatrix}-5\\-7\\7\end{pmatrix}$	A1 [3]	1.1		Condone coordinates
	(b)		$\dfrac{\begin{pmatrix}2\\-5\\-3\end{pmatrix}.\begin{pmatrix}3\\2\\-2\end{pmatrix}}{\sqrt{4+25+9}\sqrt{9+4+4}}$ soi $=\dfrac{6-10+6}{\sqrt{38}\sqrt{17}}=\dfrac{2}{\sqrt{646}}=0.07868...$	M1	1.1	BC. Using $\cos\theta=\dfrac{\mathbf{a.b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$	May see $\sin\phi=\dfrac{\mathbf{a.b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ Or use of cross product
			$\theta=$ awrt 85.5…° soi	A1	1.1	Can be implied by correct final answer	or 1.49… rads
			($\phi=90°-85.48…°=$) awrt 4.51°	A1 [3]	1.1		or 0.0788 rads

Question			Answer	Marks	AO	Guidance	
3	(c)		$\lambda = 1 \Rightarrow \mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$	B1	3.1a		
			$\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} = \begin{pmatrix} -16 \\ 5 \\ -19 \end{pmatrix}$	M1	2.2a	Method shown or at least two terms correctly evaluated	
			So equation of l_2 is $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -16 \\ 5 \\ -19 \end{pmatrix}$ oe	A1	1.1	Must be $\mathbf{r} =$. Allow parameter λ.	
				[3]			
4			DR				
			$\sum_{r=1}^{100} (2r+3)^2 = 4\sum_{r=1}^{100} r^2 + 12\sum_{r=1}^{100} r + 9\sum_{r=1}^{100} 1$	B1	3.1a	Expanding and separating	
			$\sum_{r=1}^{100} r^2 = \frac{1}{6} \times 100(100+1)(2\times 100+1)$	M1	1.1a	Use of formula for $\sum_{r=1}^{100} r^2$	
			$4\times 338350 + 12\times \frac{1}{2} \times 100 \times 101 + 900 = 1414900$	A1	1.1		
				[3]			

Question			Answer	Marks	AO	Guidance	
5	(a)		DR				
			RHS = $2\cosh^2 x - 1 = 2\left(\frac{e^{x}+e^{-x}}{2}\right)^2 - 1$	M1	2.1	Uses correct exponential form in an attempt at proof	
			$= 2\left(\frac{e^{2x}+2+e^{-2x}}{4}\right) \; 1 \quad \frac{e^{2x}+2+e^{-2x}}{2}$				
			$= \frac{e^{2x}+e^{-2x}}{2} = \cosh 2x = \text{LHS}$	A1	2.1	AG	Proof must be complete
				[2]			
	(b)		DR				
			$2\cosh^2 x - 1 = 3\cosh x + 1$ $\Rightarrow 2\cosh^2 x - 3\cosh x - 2 = 0$	M1	3.1a	Use of identity in **(a)** to leave a three term quadratic equation in just $\cosh x$	
			$(2\cosh x + 1)(\cosh x - 2) = 0$	M1	1.1	Attempt to solve eg $\frac{-(-3)\pm\sqrt{(-3)^2-4\times 2\times(-2)}}{2\times 2}$	or $2\left(\cosh x - \frac{3}{4}\right)^2 - \frac{9}{8} - 2 = 0$
			$\cosh x = 2$ or $-½$	A1	1.1		Or solves quadratic BC
			$\cosh x \geq 1$ so $\neq -½$	A1	2.3	Justification must be seen and must contain no incorrect statements	
			$x = \cosh^{-1} 2 = \ln\left(2+\sqrt{3}\right)$	A1	1.1	For either correct answer seen	
			$x = \ln\left(2-\sqrt{3}\right)$	A1	1.1	Both correct values for x	Or $x = \ln\left(2 \quad \sqrt{3}\right)$ Mark final answer
				[6]			

			Answer	Marks	AO	Guidance	
6	**(a)**		DR A shear which leaves the x-axis invariant and which transforms the point (0, 1) to the point (2, 1).	**B1** **[1]**	2.2a	Or any useful point transformed to its image	not "scale factor" or sf
	(b)		DR det**A** $= 1\times1 - 0\times2 = 1$ and this is the area scale factor	**B1** **[1]**	2.4	Both	Detailed calculation must be shown
	(c)		DR $\begin{pmatrix}1&0\\3&1\end{pmatrix}$ seen	**B1**	3.1a		
			$\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\3&1\end{pmatrix}=\begin{pmatrix}7&2\\3&1\end{pmatrix}$	**B1**	1.1	BC	
			$\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}7&2\\3&1\end{pmatrix}$	**M1**	1.1	Correct form for stretch multiplied into their matrix in either order	
			$=\begin{pmatrix}7&2\\3p&p\end{pmatrix} \Rightarrow p = 7$	**A1** **[4]**	1.1	Correct multiplication	

Question			Answer	Marks	AO	Guidance	
7	(a)		DR $\dfrac{x^3+x^2+9x-1}{x^3+x^2+4x+4} \equiv \dfrac{x^3+x^2+4x+4+5x-5}{x^3+x^2+4x+4}$ $=1+\dfrac{5x-5}{x^3+x^2+4x+4}$ So $A = 1$, $B = 5$ and $C = -5$	B1 **[1]**	3.1a	Attempt to divide out improper fraction. Could be by symbolic division or other valid method (eg comparing coefficients or substitution of values for x)	Allow embedded answers
	(b)		DR $x^3+x^2+4x+4=(x+1)(x^2+4)$	B1	3.1a	Correct factorisation of cubic seen in working	Could be from improper fraction
			$\dfrac{5x-5}{x^3+x^2+4x+4}=\dfrac{D}{x+1}+\dfrac{Ex+F}{x^2+4}$ $D(x^2+4)+(x+1)(Ex+F)=5x-5$	M1	1.2	Correct form for partial fractions equated to their remainder rational fraction from (a). Follow through their division and factorisation.	
			$x=-1 \Rightarrow 5D=-10 \Rightarrow D=-2$	A1	1.1	Or equivalent to find D correctly.	
			$x=0 \Rightarrow -2-F=-5 \Rightarrow F=3$	A1	1.1		Allow ft A1 for second and third coefficients found.
			$x^2: D+E=0 \Rightarrow E=2$ $1-\dfrac{2}{x+1}+\dfrac{2x+3}{x^2+4}$	A1 **[5]**	1.1		Or $1-\dfrac{2}{x+1}+\dfrac{2x}{x^2+4}+\dfrac{3}{x^2+4}$

Question			Answer	Marks	AO	Guidance	
7	(c)		DR $\int_0^2 \frac{x^3+x^2+9x-1}{x^3+x^2+4x+4}\,dx = \int_0^2 1-\frac{2}{x+1}+\frac{2x}{x^2+4}+\frac{3}{x^2+4}\,dx$	*M1	3.1a	Split term with x^2+4 in denominator and $ax+b$ in numerator	
			$=\left[x-2\ln(x+1)+\ln(x^2+4)+\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$	dep*M1	1.1	Correctly integrate *their* expression (ignore limits)	
			$\left(2-2\ln 3+\ln 8+\frac{3\pi}{8}\right)-\ln 4$	M1	1.1	Correctly substitute limits to produce exact values and evaluate their $\tan^{-1}$ term	
			$2+\ln\left(\frac{2}{9}\right)+\frac{3}{8}\pi$	A1 [4]	1.1	$a=2$, $b=\frac{2}{9}$, $c=\frac{3}{8}$	

8	(a)		$F = ma = 2\dfrac{\mathrm{d}v}{\mathrm{d}t} = 4\mathrm{e}^{-2t} - kv$	M1	3.3	Use of NII with m and a replaced and with 2 forces, the given force and kv	$F=ma$ can be implicit here
			$t = \ln 2$, $v = 0.5$, $F = 0 \Rightarrow 0 = 1 - 0.5k$	M1	2.2a	Use of given conditions to derive an equation in k	Can be done first
			$k = 2 \Rightarrow 2\dfrac{\mathrm{d}v}{\mathrm{d}t} = 4\mathrm{e}^{-2t} - 2v \Rightarrow \dfrac{\mathrm{d}v}{\mathrm{d}t} + v = 2\mathrm{e}^{-2t}$	A1 **[3]**	1.1	AG	Complete argument including $F=ma$
	(b)		$\mathrm{IF} = \mathrm{e}^{\int 1\mathrm{d}t} = \mathrm{e}^t$	*B1	1.1		Or CF
			$\mathrm{e}^t\dfrac{\mathrm{d}v}{\mathrm{d}t} + \mathrm{e}^t v = \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t v\right) = \mathrm{e}^t \times 2\mathrm{e}^{-2t}$	*M1	1.1	Multiplying by IF and writing LHS as an exact derivative	Or subst correct PI into DE
			$\mathrm{e}^t v = \int \cancel{2}\mathrm{e}^{-t}\,\mathrm{d}t \mathrel{=\!\!\!\!\!+} \quad 2\mathrm{e}^{-t} \quad c$	A1	1.1	"+ c" required	Or GS $v = Ae^{-t} - 2e^{-2t}$
			$t = 0$, $v = 0 \Rightarrow c = 2$	dep*M1	3.4	Use of initial conditions to derive a value for c	Or using alternative boundary condition
			$v = 2\mathrm{e}^{-t} - 2\mathrm{e}^{-2t}$	A1 **[5]**	3.4		
	(c)		As $t \to \infty$, $v \to 0$ So speed starts at 0 and ends at 0 (and is continuous and positive between) so must reach a maximum somewhere in $t > 0$	M1 A1 **[2]**	3.4 2.4		
	(d)		v is max when $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 0$ so $t = \ln 2$	M1	2.2a	Deducing time when v is maximum	Or by finding expression for $\dfrac{\mathrm{d}v}{\mathrm{d}t}$ and solving $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 0$
			So $v_{\max} = 0.5$ (given) (or $v_{\max} = 2\mathrm{e}^{-\ln 2} - 2\mathrm{e}^{-2\ln 2} = 1 - \dfrac{2}{4} = \dfrac{1}{2}$)	A1 **[2]**	3.4		

Question			Answer	Marks	AO	Guidance	
8	(e)		$v=\dfrac{\mathrm{d}x}{\mathrm{d}t}=2\mathrm{e}^{-t}-2\mathrm{e}^{-2t}\Rightarrow x=-2\mathrm{e}^{-t}+\mathrm{e}^{-2t}+d$	M1	3.3	Integrating to find expression for x	
			$t=0,\ x=0\Rightarrow 0=-2+1+d\Rightarrow d=1$	M1	3.3	Using initial conditions to find value of (new) constant	Or definite integral with correct lower limit..
			$0.9=\ 2\mathrm{e}^{-t}\quad \mathrm{e}^{-2t}\quad 1$	M1	3.5a	Recognising that the model is only valid when x lies between 0 and 0.9	…and upper limit
			$\left(\mathrm{e}^{-t}\right)^2-2\mathrm{e}^{-t}+0.1=0\Rightarrow \mathrm{e}^{-t}=\dfrac{10\pm3\sqrt{10}}{10}$ $\Rightarrow t=\ln\left(\dfrac{10}{10-3\sqrt{10}}\right)\quad 2.97\text{ (3 sf)}$	A1 [4]	2.3	Rejecting $t=\ln\left(\dfrac{10}{10+3\sqrt{10}}\right)\quad 0$ (can be implicit)	
9	(a)		$\mathbf{A}^2=\begin{pmatrix}4&12\\0&4\end{pmatrix},\ \mathbf{A}^3\ \begin{pmatrix}8&36\\0&8\end{pmatrix},$			BC	
			$\mathbf{A}^4=\begin{pmatrix}16&96\\0&16\end{pmatrix}$	B1	2.2a		
			Conjecture: $\mathbf{A}^n=\begin{pmatrix}2^n&3n\times2^{n-1}\\0&2^n\end{pmatrix}$	B1 [2]	2.2b	Allow this mark for any conjecture which works for n = 1, 2, 3 and 4.	

Question			Answer	Marks	AO	Guidance	
9	(b)		Basis case: $n=1$: $\mathbf{A}^1 = \begin{pmatrix} 2^1 & 3\times1\times2^0 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \mathbf{A}$ so true for $n=1$	B1	2.1	Allow this mark even if the conjecture is wrong, provided that it works for $n=1$	
			Assume true for $n=k$ ie $\mathbf{A}^k = \begin{pmatrix} 2^k & 3k\times2^{k-1} \\ 0 & 2^k \end{pmatrix}$	M1	2.1	Must have statement in terms of some other variable than n. Conjecture need not be correct.	
			$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \begin{pmatrix} 2^k & 3k\times2^{k-1} \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$ $= \begin{pmatrix} 2^{k+1} & 3\times2^k+3k\times2^k \\ 0 & 2^{k+1} \end{pmatrix}$ $= \begin{pmatrix} 2^{k+1} & 3(k+1)\times2^k \\ 0 & 2^{k+1} \end{pmatrix}$	M1	2.2a	Uses inductive hypothesis properly & expands	
			So true for $n=k \Rightarrow$ true for $n=k+1$. But true for $n=1$. So true for all positive integer n	A1	2.4	AG. Manipulating terms correctly and convincingly to obtain required form. Some intermediate working must be seen and a clear conclusion must be given for the induction process.	A formal proof by induction is required for full marks.
				[4]			

Question			Answer	Marks	AO	Guidance	
10	(a)		DR $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = (1+x) + \left(\frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)$ $x > 0 \Rightarrow \frac{x^2}{2!} + \frac{x^3}{3!} + ... > 0$ $\Rightarrow e^x > 1 + x$	M1 A1 **[2]**	1.1 2.2a	Quoting and *using* the Maclaurin series AG. Result with sufficient justification	
	(b)		DR $t = x+1 \Rightarrow e^{t-1} > t \Rightarrow \frac{e^t}{e} > t \Rightarrow e^t > et$	B1 **[1]**	3.1a	AG	
	(c)		DR $t = \frac{\pi}{e} > 1$ since $2 < e < 3$ and $\pi > 3$ $e^{\frac{\pi}{e}} > e \times \frac{\pi}{e} (= \pi)$ $\Rightarrow e^\pi > \pi^e$ (ie e^π is greater)	B1 M1 A1	3.1a 3.1a 1.1	Some justification that $t > 1$ is required Substituting their choice into the inequality Answer without use of inequality in part **(b)** scores **M0A0**	
			Alternative method $t = \ln \pi$ $e^{\ln \pi} > e \ln \pi$ $e^{\ln \pi} > e \ln \pi$ $\pi > \ln(\pi^e)$ $e^\pi > \pi^e$	B1 M1 A1		Some justification that $t > 1$ is required	
				[3]			

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