OCR RECOGNISING ACHIEVEMENT

**ADVANCED GCE UNIT** **4754(A)/01**

**MATHEMATICS (MEI)**

Applications of Advanced Mathematics (C4)

**Paper A**

**THURSDAY 14 JUNE 2007**

Afternoon
Time: 1 hour 30 minutes

Additional materials:
Answer booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

**INSTRUCTIONS TO CANDIDATES**

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.

**ADVICE TO CANDIDATES**

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

**NOTE**

- This paper will be followed by **Paper B: Comprehension**.

This document consists of **6** printed pages and **2** blank pages.

## Section A (36 marks)

**1** Express $\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R$ and $\alpha$ are constants to be determined, and $0° < \alpha < 90°$.

Hence solve the equation $\sin\theta - 3\cos\theta = 1$ for $0° \leqslant \theta \leqslant 360°$. [7]

**2** Write down normal vectors to the planes $2x + 3y + 4z = 10$ and $x - 2y + z = 5$.

Hence show that these planes are perpendicular to each other. [4]

**3** Fig. 3 shows the curve $y = \ln x$ and part of the line $y = 2$.

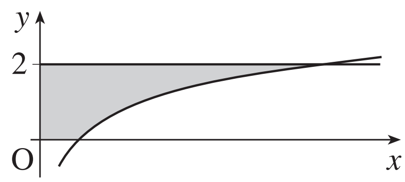


**Fig. 3**

The shaded region is rotated through 360° about the $y$-axis.

**(i)** Show that the volume of the solid of revolution formed is given by $\int_0^2 \pi e^{2y}\,dy$. [3]

**(ii)** Evaluate this, leaving your answer in an exact form. [3]

**4** A curve is defined by parametric equations

$$x = \frac{1}{t} - 1,\quad y = \frac{2+t}{1+t}.$$

Show that the cartesian equation of the curve is $y = \dfrac{3+2x}{2+x}$. [4]

**5** Verify that the point $(-1, 6, 5)$ lies on both the lines

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

Find the acute angle between the lines. [7]

**6** Two students are trying to evaluate the integral $\int_1^2 \sqrt{1+e^{-x}}\,dx$.

Sarah uses the trapezium rule with 2 strips, and starts by constructing the following table.

| $x$ | 1 | 1.5 | 2 |
|---|---|---|---|
| $\sqrt{1+e^{-x}}$ | 1.1696 | 1.1060 | 1.0655 |

**(i)** Complete the calculation, giving your answer to 3 significant figures. [2]

Anish uses a binomial approximation for $\sqrt{1+e^{-x}}$ and then integrates this.

**(ii)** Show that, provided $e^{-x}$ is suitably small, $\left(1+e^{-x}\right)^{\frac{1}{2}} \approx 1+\frac{1}{2}e^{-x}-\frac{1}{8}e^{-2x}$. [3]

**(iii)** Use this result to evaluate $\int_1^2 \sqrt{1+e^{-x}}\,dx$ approximately, giving your answer to 3 significant figures. [3]

**Section B** (36 marks)

**7** Data suggest that the number of cases of infection from a particular disease tends to oscillate between two values over a period of approximately 6 months.

**(a)** Suppose that the number of cases, $P$ thousand, after time $t$ months is modelled by the equation $P = \dfrac{2}{2 - \sin t}$. Thus, when $t = 0, P = 1$.

**(i)** By considering the greatest and least values of $\sin t$, write down the greatest and least values of $P$ predicted by this model. [2]

**(ii)** Verify that $P$ satisfies the differential equation $\dfrac{dP}{dt} = \frac{1}{2} P^2 \cos t$. [5]

**(b)** An alternative model is proposed, with differential equation

$$\frac{dP}{dt} = \frac{1}{2}(2P^2 - P)\cos t. \qquad (*)$$

As before, $P = 1$ when $t = 0$.

**(i)** Express $\dfrac{1}{P(2P-1)}$ in partial fractions. [4]

**(ii)** Solve the differential equation (*) to show that

$$\ln\left(\frac{2P-1}{P}\right) = \frac{1}{2}\sin t. \qquad [5]$$

This equation can be rearranged to give $P = \dfrac{1}{2 - e^{\frac{1}{2}\sin t}}$.

**(iii)** Find the greatest and least values of $P$ predicted by this model. [4]

**8**

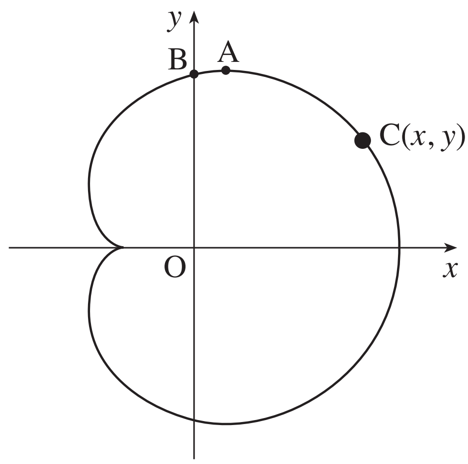


**Fig. 8**

In a theme park ride, a capsule C moves in a vertical plane (see Fig. 8). With respect to the axes shown, the path of C is modelled by the parametric equations

$$x = 10\cos\theta + 5\cos 2\theta, \quad y = 10\sin\theta + 5\sin 2\theta, \qquad (0 \leqslant \theta < 2\pi),$$

where $x$ and $y$ are in metres.

**(i)** Show that $\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$.

Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \frac{1}{3}\pi$. Hence find the exact coordinates of the highest point A on the path of C. [6]

**(ii)** Express $x^2 + y^2$ in terms of $\theta$. Hence show that

$$x^2 + y^2 = 125 + 100\cos\theta. \qquad [4]$$

**(iii)** Using this result, or otherwise, find the greatest and least distances of C from O. [2]

You are given that, at the point B on the path vertically above O,

$$2\cos^2\theta + 2\cos\theta - 1 = 0.$$

**(iv)** Using this result, and the result in part (**ii**), find the distance OB. Give your answer to 3 significant figures. [4]

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RECOGNISING ACHIEVEMENT

**ADVANCED GCE UNIT** **4754(B)/01**

## MATHEMATICS (MEI)

Applications of Advanced Mathematics (C4)

**Paper B: Comprehension**

**THURSDAY 14 JUNE 2007**

Afternoon
Time: Up to 1 hour

Additional materials:
Rough paper
MEI Examination Formulae and Tables (MF2)

Candidate Name

Centre Number

Candidate Number

**INSTRUCTIONS TO CANDIDATES**

- Write your name, centre number and candidate number in the spaces above.
- Answer **all** the questions.
- Write your answers in the spaces provided on the question paper.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

**INFORMATION FOR CANDIDATES**

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 18.
- The insert contains the text for use with the questions.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with the question paper.

**ADVICE TO CANDIDATES**

- Read each question carefully and make sure you know what you have to do before starting your answer.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

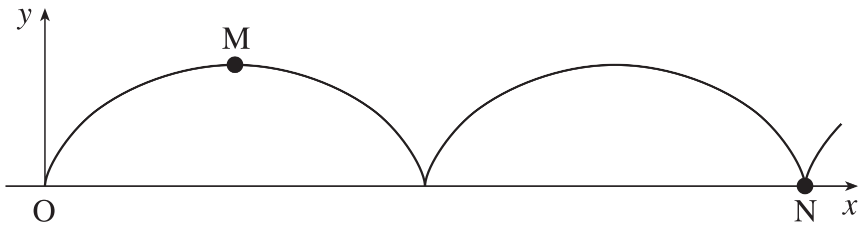
| For Examiner's Use | |
|---|---|
| Qu. | Mark |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| Total | |

This document consists of **5** printed pages, **3** blank pages and an insert.

For Examiner's Use

**1** This basic cycloid has parametric equations

$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta.$$




Find the coordinates of the points M and N, stating the value of $\theta$ at each of them. [2]

Point M ..............................................................................................................................

Point N ..............................................................................................................................

**2** A sea wave has parametric equations (in suitable units)

$$x = 7\theta - 0.25\sin\theta, \quad y = 0.25\cos\theta.$$

Find the wavelength and height of the wave. [3]

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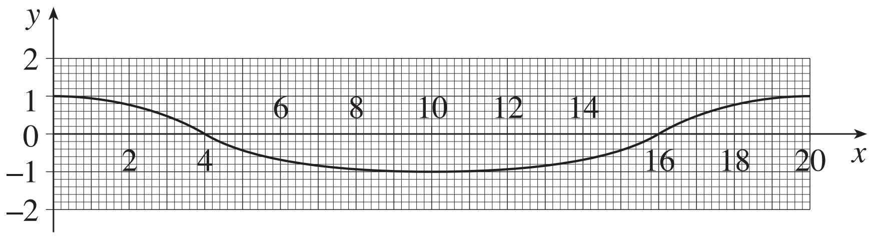
For Examiner's Use

**3** The graph below shows the profile of a wave.

**(i)** Assuming that it has parametric equations of the form given on line 68, find the values of $a$ and $b$. [2]

**(ii)** Investigate whether the ratio of the trough length to the crest length is consistent with this shape. [3]




**(i)** ........................................................................................................................................

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**(ii)** ........................................................................................................................................

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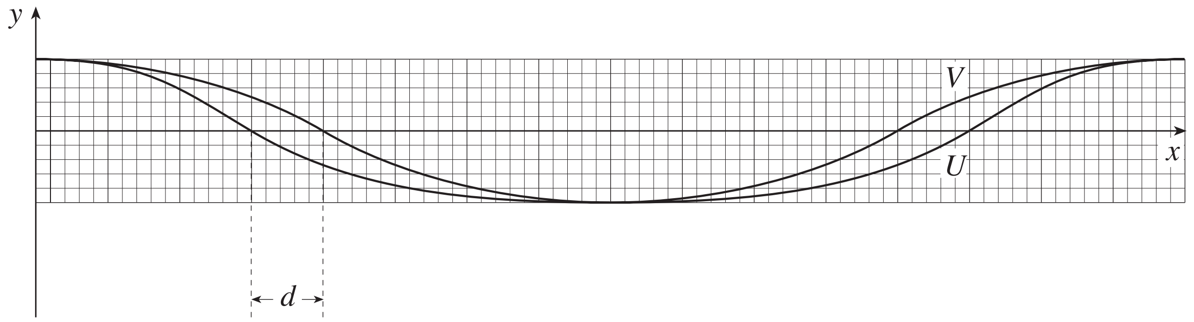
**4** This diagram illustrates two wave shapes $U$ and $V$. They have the same wavelength and the same height.




One of the curves is a sine wave, the other is a curtate cycloid.

**(i)** State which is which, justifying your answer. [1]

**(i)** ..............................................................................................................................

..............................................................................................................................

The parametric equations for the curves are:

$$x = a\theta, \quad y = b\cos\theta,$$

and

$$x = a\theta - b\sin\theta, \quad y = b\cos\theta.$$

**(ii)** Show that the distance marked $d$ on the diagram is equal to $b$. [2]

**(iii)** Hence justify the statement in lines 109 to 111: "In such cases, the curtate cycloid and the sine curve with the same wavelength and height are very similar and so the sine curve is also a good model." [2]

**(ii)** ..............................................................................................................................

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**(iii)** ..............................................................................................................................

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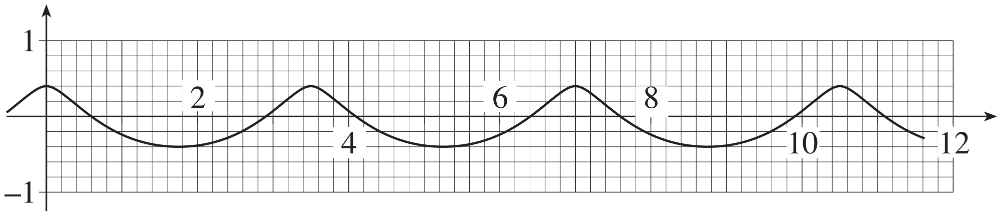
For Examiner's Use

**5** The diagram shows a curtate cycloid with scales given. Show that this curve could not be a scale drawing of the shape of a stable sea wave. [3]

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RECOGNISING ACHIEVEMENT

**ADVANCED GCE UNIT**

**MATHEMATICS (MEI)**

Applications of Advanced Mathematics (C4)

**Paper B: Comprehension**

**INSERT**

**THURSDAY 14 JUNE 2007**

# 4754(B)/01

Afternoon
Time: Up to 1 hour

**INSTRUCTIONS TO CANDIDATES**

- This insert contains the text for use with the questions.

This document consists of **8** printed pages.

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# Modelling sea waves

## Introduction

There are many situations in which waves and oscillations occur in nature and often they are accurately modelled by the sine curve. However, this is not the case for sea waves as these come in a variety of shapes. The photograph in Fig. 1 shows an extreme form of sea wave being ridden by a surfer.

**Fig. 1**

At any time many parts of the world's oceans are experiencing storms. The strong winds create irregular *wind waves*. However, once a storm has passed, the waves form into a regular pattern, called *swell*. Swell waves are very stable; those resulting from a big storm would travel several times round the earth if they were not stopped by the land.

Fig. 2 illustrates a typical swell wave, but with the vertical scale exaggerated. The horizontal distance between successive *peaks* is the *wavelength*; the vertical distance from the lowest point in a *trough* to a peak is called the *height*. The height is twice the *amplitude* which is measured from the horizontal *line of mid-height*. The upper part is the *crest*. These terms are illustrated in Fig. 2.

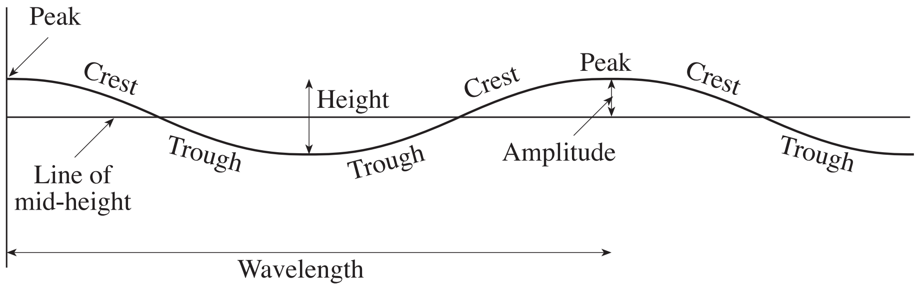


**Fig. 2**

The speed of a wave depends on the depth of the water; the deeper the water, the faster the wave. (This is, however, not true for very deep water, where the wave speed is independent of the depth.) This has a number of consequences for waves as they come into shallow water.

- Their speed decreases.
- Their wavelength shortens.
- Their height increases.

Observations show that, as their height increases, the waves become less symmetrical. The troughs become relatively long and the crests short and more pointed.

The profile of a wave approaching land is illustrated in Fig. 3. Eventually the top curls over and the wave “breaks”.

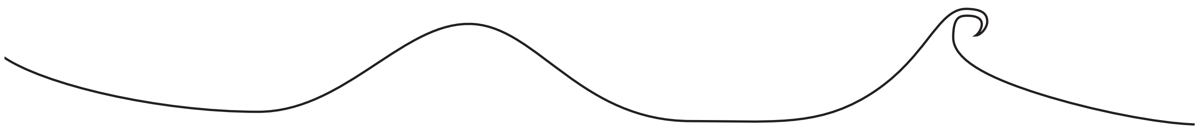

**Fig. 3**

If you stand at the edge of the sea you will see the water from each wave running up the shore towards you. You might think that this water had just travelled across the ocean. That would be wrong. When a wave travels across deep water, it is the shape that moves across the surface and not the water itself. It is only when the wave finally reaches land that the actual water moves any significant distance.

Experiments in wave tanks have shown that, except near the shore, each drop of water near the surface undergoes circular motion (or very nearly so). This has led people to investigate the possiblility that a form of cycloid would provide a better model than a sine curve for a sea wave.

## Cycloids

There are several types of cycloid. In this article, the name *cycloid* refers to one of the family of curves which form the locus of a point on a circle rolling along a straight horizontal path.

Fig. 4 illustrates the basic cycloid; in this case the point is on the circumference of the circle.

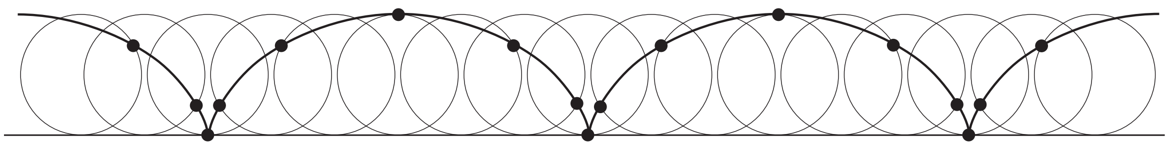

**Fig. 4**

Two variations on this basic cycloid are the prolate cycloid, illustrated in Fig. 5, and the curtate cycloid illustrated in Fig. 6. The prolate cycloid is the locus of a point attached to the circle but outside the circumference (like a point on the flange of a railway train's wheel); the curtate cycloid is the locus of a point inside the circumference of the circle.

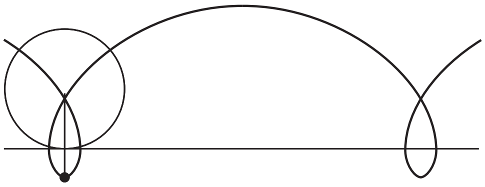

**Fig. 5**

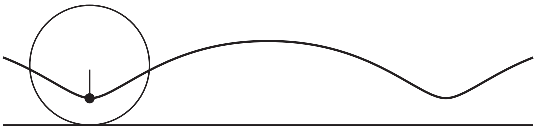

**Fig. 6**

When several cycles of the curtate cycloid are drawn "upside down", as in Fig. 7, the curve does indeed look like the profile of a wave in shallow water.

**Fig. 7**

**The equation of a cycloid**

The equation of a cycloid is usually given in parametric form.

Fig. 8.1 and Fig. 8.2 illustrate a circle rolling along the $x$-axis. The circle has centre Q and radius $a$. P and R are points on its circumference and angle PQR = $\theta$, measured in radians. Fig. 8.1 shows the initial position of the circle with P at its lowest point; this is the same point as the origin, O. Some time later the circle has rolled to the position shown in Fig. 8.2 with R at its lowest point.

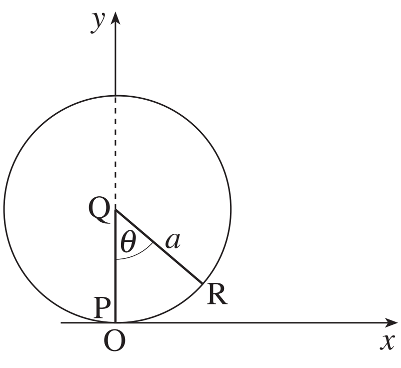


**Fig. 8.1**

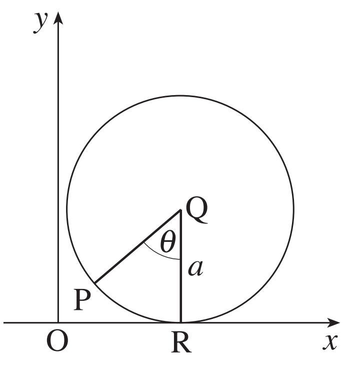


**Fig. 8.2**

In travelling to its new position, the circle has rolled the distance OR in Fig. 8.2. Since it has rolled along its circumference, this distance is the same as the arc length PR, and so is $a\theta$. Thus the coordinates of the centre, Q, in Fig. 8.2 are $(a\theta, a)$. To find the coordinates of the point P in Fig. 8.2, look at triangle QPZ in Fig. 9.

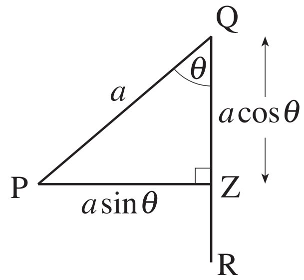


**Fig. 9**

You can see that

$$PZ = a\sin\theta \qquad \text{and} \qquad QZ = a\cos\theta.$$

Hence the coordinates of P are $(a\theta - a\sin\theta, a - a\cos\theta)$, and so the locus of the point P is described by the curve with parametric equations

$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta.$$

This is the basic cycloid.

These parametric equations can be generalised to

$$x = a\theta - b\sin\theta, \quad y = a - b\cos\theta,$$

where $b$ is the distance of the moving point from the centre of the circle.

For $b < a$ the curve is a *curtate cycloid*,
$b = a$ the curve is a *basic cycloid*,
$b > a$ the curve is a *prolate cycloid*.

The equivalent equations with the curve turned "upside down", and with the mid-height of the curve now on the $x$-axis, are

$$x = a\theta - b\sin\theta, \quad y = b\cos\theta.$$

(Notice that positive values of $y$ are still measured vertically upwards.)

## Modelling a particular wave

A question that now arises is how to fit an equation to a particular wave profile.

If you assume that the wave is a cycloid, there are two parameters to be found, $a$ and $b$.

Since $y = b\cos\theta$,

- the maximum value of $y$ is $b$ and this occurs when $\theta = 0, 2\pi, 4\pi, \ldots$ ,
- the minimum value of $y$ is $-b$ and this occurs when $\theta = \pi, 3\pi, 5\pi, \ldots$ ,
- the height of the wave is $2b$.

The wavelength is the horizontal distance between successive maximum values of $y$.

- A maximum occurs when $\theta = 0$ and $x = 0$.
- The next maximum occurs when $\theta = 2\pi$ and in that case $x = a \times (2\pi) - b \times \sin(2\pi) = 2\pi a$.
- The wavelength is $2\pi a$.

Thus if a wave has a cycloid form, $a = \dfrac{\text{wavelength}}{2\pi}$ and $b = \dfrac{\text{height}}{2}$.

The profile of a possible wave shape is illustrated in Fig. 10. You can see that the wavelength is 16 units and the height is 2 units. So if it is a cycloid shape, the values of $a$ and $b$ would be 2.54… and 1; it is actually easier to work with $\pi a$, which would have the value 8 in this case.

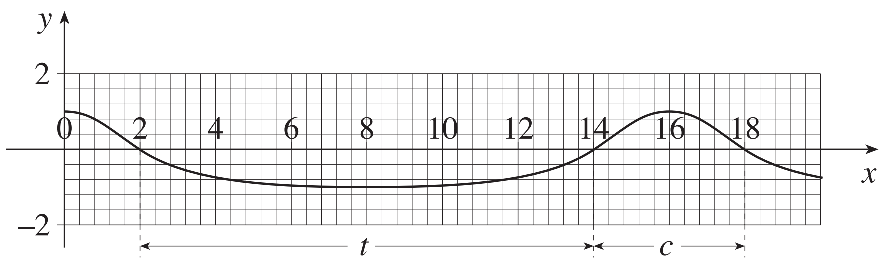


**Fig. 10**

However, finding values for $a$ and $b$ does not in itself show that the form of a wave is indeed a cycloid. One way of checking whether this could be a good model is to measure the length of the trough of the wave (the distance for which it is below mid-height, and so $y < 0$) and the length of its crest (the distance for which it is above mid-height, and so $y > 0$.) These distances are marked as $t$ and $c$ in Fig. 10.

The ratio of the measured distances $t$ and $c$ is then compared with the equivalent ratio for a cycloid.

In Fig. 10, $t = 12$ and $c = 4$, so the ratio $t : c$ is $3 : 1$.

To find the equivalent ratio for a cycloid, start by finding the values of $\theta$ for which the wave is at mid-height.

When $y = 0$, the values of $\theta$ are $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots$ .

 

The corresponding values of $x$ are $\left(\frac{\pi a}{2}-b\right)$, $\left(\frac{3\pi a}{2}+b\right)$, $\left(\frac{5\pi a}{2}-b\right)$, ….

So
$$t=\left(\frac{3\pi a}{2}+b\right)-\left(\frac{\pi a}{2}-b\right)=\pi a+2b,$$
$$c=\left(\frac{5\pi a}{2}-b\right)-\left(\frac{3\pi a}{2}+b\right)=\pi a-2b,$$

and the ratio $t:c$ is $(\pi a+2b):(\pi a-2b)$.

Using $\pi a = 8$ and $b = 1$, the values of $t$ and $c$ would be 10 and 6 and the ratio $t:c$ would be $10:6$ or $1.67:1$.

As this ratio is quite different from 3 : 1, the curve in Fig. 10 is not a cycloid. In this case, the troughs are too long and the crests too short.

**Sea waves**

In fact, observations of real swell waves show that they are well modelled as curtate cycloids.

In the deep ocean, the wavelength may be hundreds of metres and the height less than 5 metres. This corresponds to a large value of $a$ and a small value of $b$. For a wavelength of 200 metres and a height of 2 metres, the ratio of height to wavelength is 1 : 100. In such cases, the curtate cycloid and the sine curve with the same wavelength and height are very similar and so the sine curve is also a good model.

As the wave comes into shallower water, the ratio of height to wavelength increases. The curtate cycloid remains a good model but the sine curve becomes increasingly unsuitable.

During this phase the value of $a$ decreases and the value of $b$ increases. Thought of in terms of the locus of a point on a rolling circle, the point moves away from the centre while the radius decreases.

Eventually, however, the wave breaks with its top curling over. It is tempting to imagine that this would correspond to the case $b = a$, when the cycloid changes from curtate to prolate and develops a loop. That would correspond to a height to wavelength ratio of $1:\pi$. However, observation shows that breaking occurs well before that, when the height to wavelength ratio is about 1 : 7.

This observation is not surprising when you remember that the motion of the drops of water in a wave is circular. Such circular motion cannot occur at the sharp point at the peak of a basic cycloid. The wave becomes unstable when the circular motion can no longer be sustained within the wave. At this point the cycloid ceases to provide a model for sea waves.



[Turn over

**Other sea waves**

In addition to swell and wind waves, there are several other types of sea wave.

*Internal waves* are formed when a current runs over an uneven seabed. Because the currents are often caused by the tides, these waves are often called tidal rips. They often form off headlands, and can also take the form of whirlpools like the famous Corrievrechan off the west of Scotland.

*Tidal waves* are caused by the combined gravitational pull of the moon and sun; they have a period of just over 12 hours and a wavelength of half the circumference of the earth. In mid-ocean their height is extremely small but in coastal waters it can be over 10 metres.

*Tsunamis* are caused by events such as earthquakes and volcanoes. Compared with swell, a tsunami has a very long wavelength, typically at least 100 kilometres. In mid-ocean their height is small and so they are not usually noticed by sailors. However, because of their long wavelength, they build up to a great height when they come into land and cause devastation to coastal areas.

The study of sea waves has been given a boost recently by satellite imaging. This allows the profiles of waves to be determined accurately. One discovery is that *very high waves* are much more common than had been expected. These giant waves are not well understood and it may well be that they require a new model.