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RECOGNISING ACHIEVEMENT

Friday 18 January 2013 – Afternoon

A2 GCE MATHEMATICS (MEI)

4754/01A Applications of Advanced Mathematics (C4) Paper A

QUESTION PAPER

Candidates answer on the Printed Answer Book.

OCR supplied materials:
- Printed Answer Book 4754/01A
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator

Duration: 1 hour 30 minutes

4734020113

INSTRUCTIONS TO CANDIDATES

These instructions are the same on the Printed Answer Book and the Question Paper.

- The Question Paper will be found in the centre of the Printed Answer Book.
- Write your name, centre number and candidate number in the spaces provided on the Printed Answer Book. Please write clearly and in capital letters.
- **Write your answer to each question in the space provided in the Printed Answer Book**. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

This information is the same on the Printed Answer Book and the Question Paper.

- The number of marks is given in brackets **[ ]** at the end of each question or part question on the Question Paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- The Printed Answer Book consists of **16** pages. The Question Paper consists of **4** pages. Any blank pages are indicated.
- This paper will be followed by **Paper B: Comprehension**.

INSTRUCTION TO EXAMS OFFICER / INVIGILATOR

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Section A (36 marks)

1 Solve the equation $\dfrac{2x}{x+1} - \dfrac{1}{x-1} = 1$. **[4]**

2 Find the first four terms of the binomial expansion of $\sqrt[3]{1-2x}$. State the set of values of x for which the expansion is valid. **[6]**

3 The parametric equations of a curve are

$$x = \sin\theta, \quad y = \sin 2\theta, \quad \text{for } 0 \leqslant \theta \leqslant 2\pi.$$

(i) Find the exact value of the gradient of the curve at the point where $\theta = \frac{1}{6}\pi$. **[4]**

(ii) Show that the cartesian equation of the curve is $y^2 = 4x^2 - 4x^4$. **[3]**

4 Fig. 4 shows the curve $y = \sqrt{1 + e^{2x}}$, and the region between the curve, the x-axis, the y-axis and the line $x = 2$.

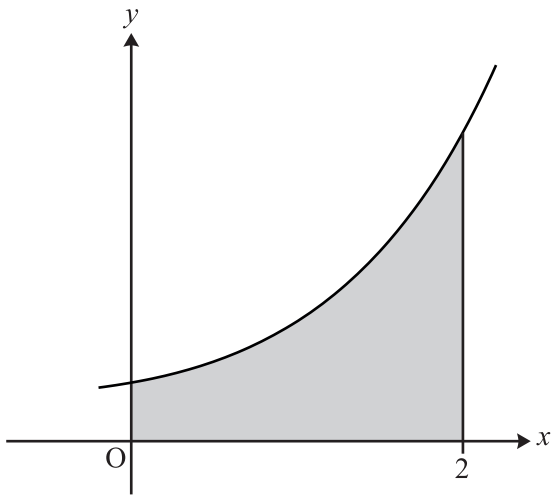

Fig. 4

(a) Find the exact volume of revolution when the shaded region is rotated through 360° about the x-axis. **[4]**

(b) **(i)** Complete the table of values, and use the trapezium rule with 4 strips to estimate the area of the shaded region. **[3]**

x	0	0.5	1	1.5	2
y		1.9283	2.8964	4.5919	

(ii) The trapezium rule for $\int_0^2 \sqrt{1 + e^{2x}}\,dx$ with 8 and 16 strips gives 6.797 and 6.823, although not necessarily in that order. Without doing the calculations, say which result is which, explaining your reasoning. **[1]**

5 Solve the equation $2\sec^2\theta = 5\tan\theta$, for $0 \leqslant \theta \leqslant \pi$. **[6]**

6 In Fig. 6, ABC, ACD and AED are right-angled triangles and BC = 1 unit. Angles CAB and CAD are θ and ϕ respectively.

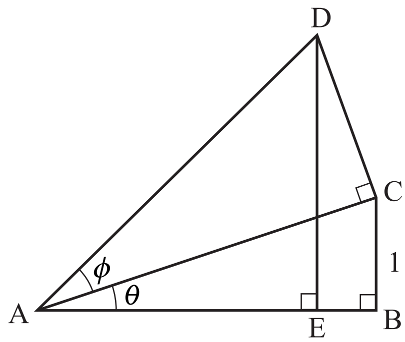

Fig. 6

(i) Find AC and AD in terms of θ and ϕ. **[2]**

(ii) Hence show that $\mathrm{DE} = 1 + \dfrac{\tan\phi}{\tan\theta}$. **[3]**

Section B (36 marks)

7 A tent has vertices ABCDEF with coordinates as shown in Fig. 7. Lengths are in metres. The Oxy plane is horizontal.

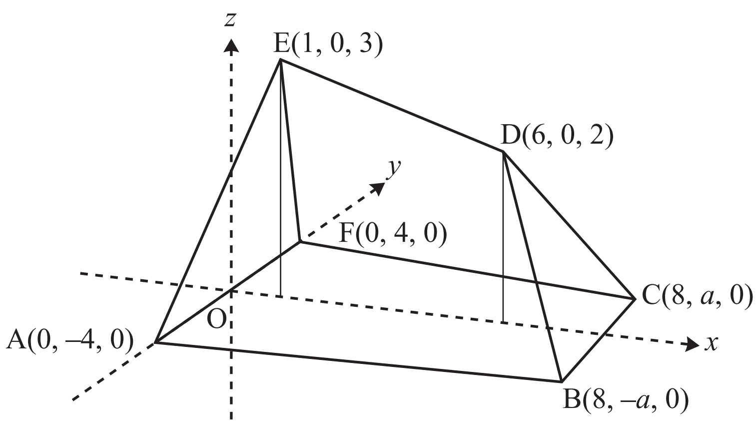

Fig. 7

(i) Find the length of the ridge of the tent DE, and the angle this makes with the horizontal. **[4]**

(ii) Show that the vector $\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ is normal to the plane through A, D and E.

Hence find the equation of this plane. Given that B lies in this plane, find a. **[7]**

(iii) Verify that the equation of the plane BCD is $x + z = 8$.

Hence find the acute angle between the planes ABDE and BCD. **[6]**

8 The growth of a tree is modelled by the differential equation

$$10\frac{\mathrm{d}h}{\mathrm{d}t} = 20 - h,$$

where h is its height in metres and the time t is in years. It is assumed that the tree is grown from seed, so that $h = 0$ when $t = 0$.

(i) Write down the value of h for which $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, and interpret this in terms of the growth of the tree. **[1]**

(ii) Verify that $h = 20\left(1 - \mathrm{e}^{-0.1t}\right)$ satisfies this differential equation and its initial condition. **[5]**

The alternative differential equation

$$200\frac{\mathrm{d}h}{\mathrm{d}t} = 400 - h^2$$

is proposed to model the growth of the tree. As before, $h = 0$ when $t = 0$.

(iii) Using partial fractions, show by integration that the solution to the alternative differential equation is

$$h = \frac{20\left(1 - \mathrm{e}^{-0.2t}\right)}{1 + \mathrm{e}^{-0.2t}}.$$

[9]

(iv) What does this solution indicate about the long-term height of the tree? **[1]**

(v) After a year, the tree has grown to a height of 2 m. Which model fits this information better? **[3]**

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A2 GCE MATHEMATICS (MEI)

4754/01B Applications of Advanced Mathematics (C4) Paper B: Comprehension

QUESTION PAPER

Candidates answer on the Question Paper.

OCR supplied materials:
- Insert (inserted)
- MEI Examination Formulae and Tables (MF2)

Other materials required:
- Scientific or graphical calculator
- Rough paper

Duration: Up to 1 hour

Candidate forename		Candidate surname	

Centre number						Candidate number				

INSTRUCTIONS TO CANDIDATES

- The insert will be found in the centre of this document.
- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Write your answer to each question in the space provided. Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- The insert contains the text for use with the questions.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You may find it helpful to make notes and do some calculations as you read the passage.
- You are **not** required to hand in these notes with your question paper.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **18**.
- This document consists of **8** pages. Any blank pages are indicated.

BLANK PAGE

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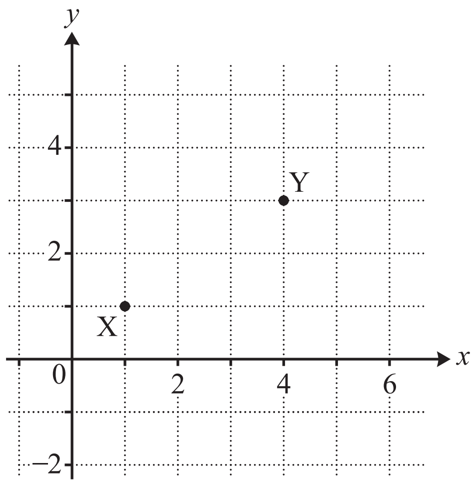
1 On the grid below mark all three possible positions of the point P with integer coordinates for which $t(P,X) = 4$ and $t(P,Y) = 3$. **[3]**

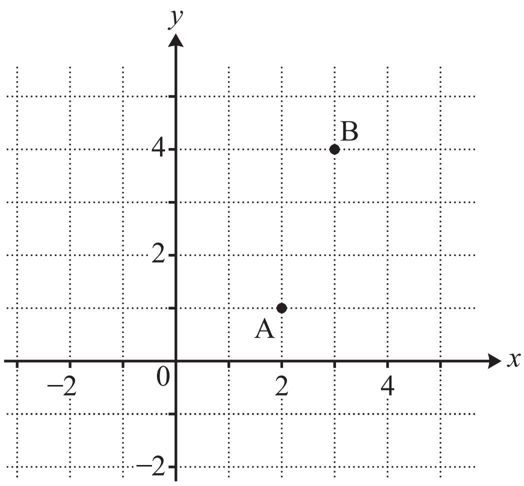
1



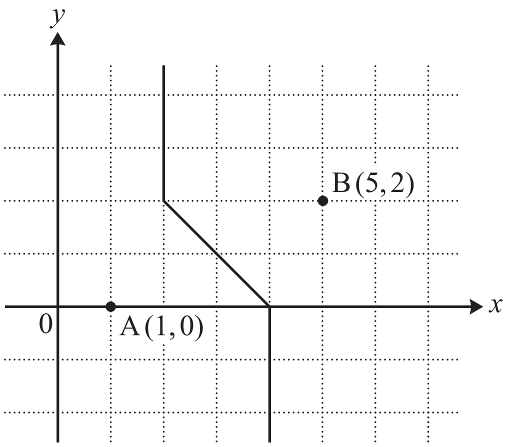
2 This question is concerned with generalised taxicab geometry.

On the grid below, show the locus of a point P where $t(P,A) = t(P,B)$. **[3]**

2

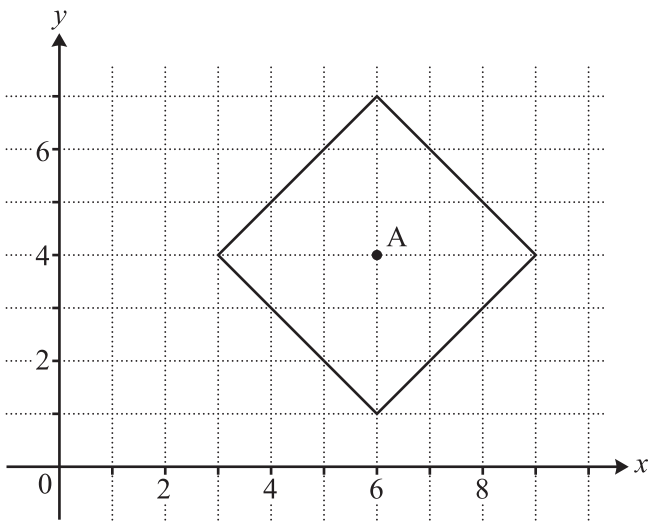


3 **(i)** Describe the following locus of a point P, using the notation $\mathrm{t}(\mathrm{P},\mathrm{A})$ and $\mathrm{t}(\mathrm{P},\mathrm{B})$ as appropriate.



[1]

(ii) Describe the following locus of a point P, using the notation $\mathrm{t}(\mathrm{P},\mathrm{A})$ as appropriate.



[1]

3 (i)	
3 (ii)	

PLEASE DO NOT WRITE IN THIS SPACE

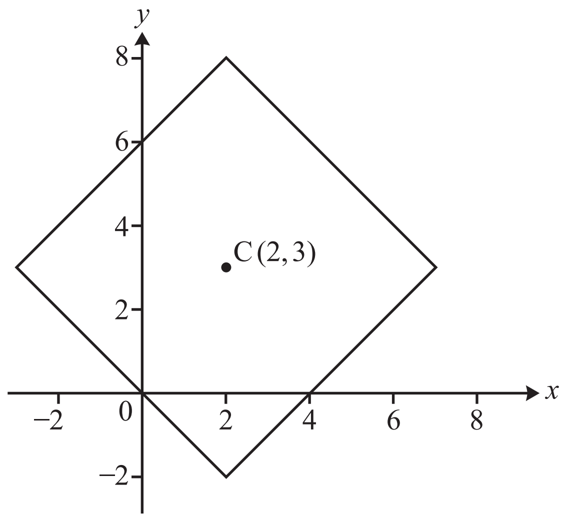
4 Referring to Fig. 5, or otherwise, find the value of $n(4,4)$. **[2]**

4	

5 In lines 54 and 55 it says there are 35 minimum distance routes from A $(0,0)$ to B $(4,3)$. Determine how many of these routes pass through the point with coordinates $(3,2)$, explaining your reasoning. **[2]**

5	

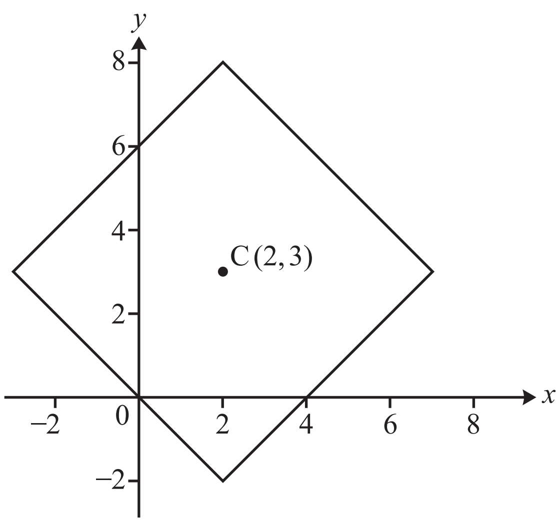
6 Fig. 7 is reproduced below.



(i) Two points on this locus have x-coordinate -0.7. Write down the coordinates of each of these points. **[2]**

(ii) In lines 77 to 78 it says "adding a second taxicab circle with centre $(2,0)$ and radius 2 shows that in generalised taxicab geometry two different circles can have an infinite number of points in common!"

On the copy of Fig. 7 given below, draw the taxicab circle with centre $(2,0)$ and radius 2. **[1]**

6 (i)	
6 (ii)	

7 In lines 23 and 24 it says that "if the Pythagorean distance between two points A and B is $\mathrm{d}(\mathrm{A},\mathrm{B})$ then the taxicab distance satisfies the inequalities $\mathrm{d}(\mathrm{A},\mathrm{B}) \leqslant \mathrm{t}(\mathrm{A},\mathrm{B}) \leqslant \sqrt{2} \times \mathrm{d}(\mathrm{A},\mathrm{B})$."

This question is about using this result in generalised taxicab geometry.

(i) Given that A is the point $(0,0)$, describe all possible positions of B for which $\mathrm{d}(\mathrm{A},\mathrm{B}) = \mathrm{t}(\mathrm{A},\mathrm{B})$. **[1]**

(ii) Given that A is the point $(0,0)$, describe all possible positions of B for which $\mathrm{t}(\mathrm{A},\mathrm{B}) = \sqrt{2} \times \mathrm{d}(\mathrm{A},\mathrm{B})$. **[2]**

7 (i)	
7 (ii)	

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A2 GCE MATHEMATICS (MEI)

4754/01B Applications of Advanced Mathematics (C4) Paper B: Comprehension

INSERT

Duration: Up to 1 hour

INFORMATION FOR CANDIDATES

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Taxicab geometry

Introduction

Fig. 1 shows part of the road map of an imaginary town called Newtown.

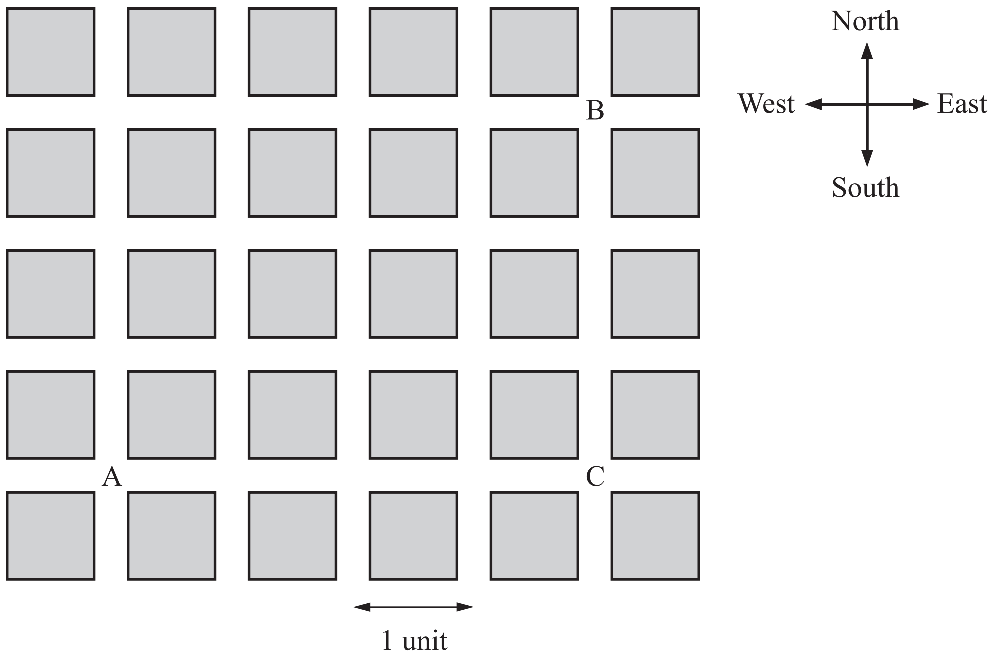

Fig. 1

Newtown's buildings are grouped in equal-sized square blocks. The roads between the blocks run in north-south and east-west directions and traffic can travel along every road in both directions.

Imagine you want to take a taxi from point A to point B. If the taxi travelled east from A to C and then north from C to B, the total distance travelled would be 7 units. Many other routes from A to B are also 7 units in length but no route is shorter. This shortest distance is called the *taxicab distance* from A to B and the related mathematics is called *taxicab geometry*.

This article introduces some of the mathematics of taxicab geometry.

Introducing the notation

Fig. 2 shows part of the road map of Newtown and one particular bus route with bus stops at positions M and N. Imagine you are at position L and you wish to catch a bus at one of these bus stops. Which is closer?

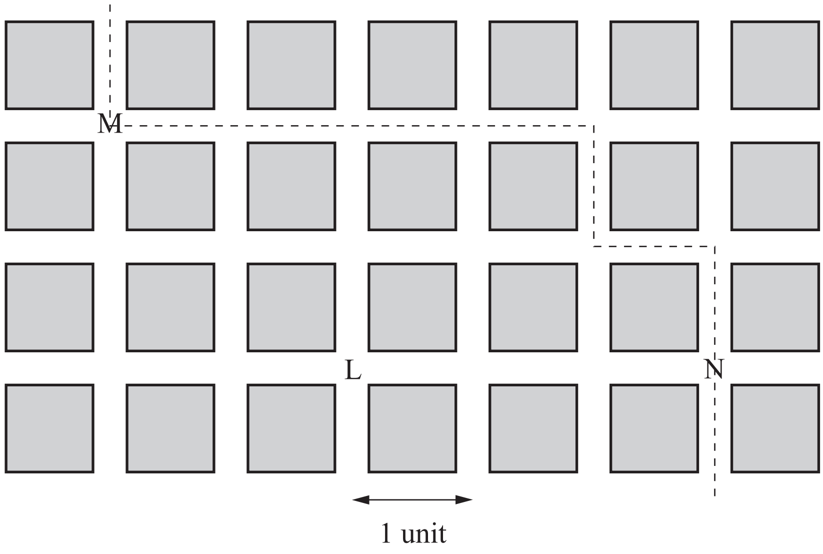

Fig. 2

By Pythagoras's Theorem, the straight line distance, measured in units as shown in Fig. 2, from L to M is $2\sqrt{2}$. This is expressed using the notation $\mathrm{d}(\mathrm{L},\mathrm{M}) = 2\sqrt{2}$. Similarly, $\mathrm{d}(\mathrm{L},\mathrm{N}) = 3$. In terms of straight line distances, M is closer than N since $\mathrm{d}(\mathrm{L},\mathrm{M}) < \mathrm{d}(\mathrm{L},\mathrm{N})$.

For a pedestrian, who is constrained to walking along roads, it is the taxicab distance rather than the Pythagorean distance that is important. The taxicab distance from L to M is 4. This is expressed using the notation $\mathrm{t}(\mathrm{L},\mathrm{M}) = 4$. Similarly $\mathrm{t}(\mathrm{L},\mathrm{N}) = 3$. For a pedestrian at L, since $\mathrm{t}(\mathrm{L},\mathrm{N}) < \mathrm{t}(\mathrm{L},\mathrm{M})$, N is closer than M.

This is an example of a situation in which $\mathrm{d}(\mathrm{L},\mathrm{M}) < \mathrm{d}(\mathrm{L},\mathrm{N})$ but $\mathrm{t}(\mathrm{L},\mathrm{M}) > \mathrm{t}(\mathrm{L},\mathrm{N})$.

In general, if the Pythagorean distance between two points A and B is $\mathrm{d}(\mathrm{A},\mathrm{B})$ then the taxicab distance satisfies the inequalities $\mathrm{d}(\mathrm{A},\mathrm{B}) \leqslant \mathrm{t}(\mathrm{A},\mathrm{B}) \leqslant \sqrt{2} \times \mathrm{d}(\mathrm{A},\mathrm{B})$.

Minimum distance routes

In Fig. 1 the Pythagorean distance between the points A and B is 5. There is only one straight line segment from A to B; its length is 5.

However, this uniqueness property does not hold when considering the taxicab distance. In Fig. 1, the taxicab distance from A to B is 7. There are several routes from A to B which have this minimum distance; these are called *minimum distance routes*.

How many minimum distance routes are there from A to B?

In order to answer this question, the road grid is replaced by a coordinate system as shown in Fig. 3. The x-axis represents the west-east direction and the y-axis represents the south-north direction. Point A has coordinates $(0,0)$ and point B has coordinates $(4,3)$. The roads are shown by the grid lines.

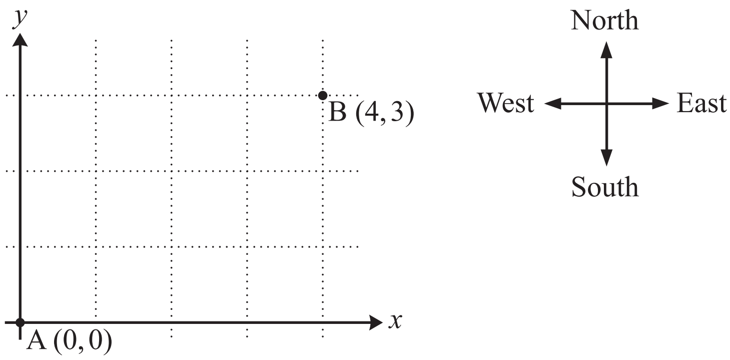

Fig. 3

Clearly, no minimum distance route from A to any point in the first quadrant will involve any motion in a westerly or southerly direction.

There is only one minimum distance route from A to any point on the x-axis or to any point on the y-axis.

There are two ways of reaching the point with coordinates $(1,1)$ along minimum distance routes as follows.

$$(0,0) \to (1,0) \to (1,1)$$

$$(0,0) \to (0,1) \to (1,1)$$

The numbers of minimum distance routes from A to the points mentioned above are shown in Fig. 4.

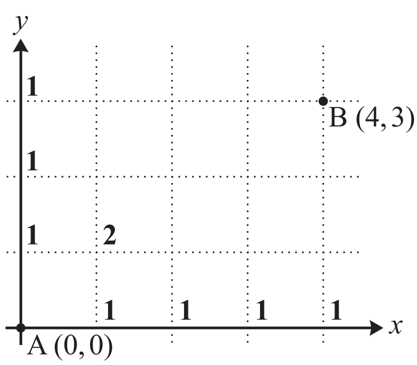

Fig. 4

The final step on a minimum distance route from A to the point $(2,1)$ must be either from $(2,0)$ to $(2,1)$ or from $(1,1)$ to $(2,1)$. There is 1 minimum distance route from A to $(2,0)$ and there are 2 minimum distance routes from A to $(1,1)$. Each of these routes can be continued to $(2,1)$ in only one way. Since all of these routes are different, the number of minimum distance routes from A to $(2,1)$ is 3.

This reasoning can be extended to other grid points. The notation $\mathrm{n}(p,q)$ is used to denote the number of minimum distance routes from $(0,0)$ to (p,q), where p and q are non-negative integers. The following rules apply for $p \geqslant 1, q \geqslant 1$.

$$\mathrm{n}(p,0) = 1$$
$$\mathrm{n}(0,q) = 1$$
$$\mathrm{n}(p,q) = \mathrm{n}(p-1,q) + \mathrm{n}(p,q-1)$$

These rules give the numbers of minimum distance routes shown in Fig. 5.

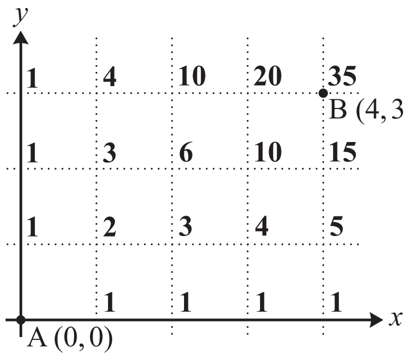

Fig. 5

So the answer to the question of how many minimum distance routes there are from $\mathrm{A}(0,0)$ to $\mathrm{B}(4,3)$ is 35.

Generalised taxicab geometry

The mathematical model of taxicab geometry described so far has been motivated by a system of roads and junctions. In this system there is a finite number of uniformly spaced parallel and perpendicular roads and all journeys start and end at junctions.

The mathematical ideas can be generalised by defining the taxicab distance for any two points in the x-y plane. In this generalised version, the points are not necessarily grid points.

Fig. 6 shows two points, $\mathrm{R}(x_1, y_1)$ and $\mathrm{S}(x_2, y_2)$, in the x-y plane.

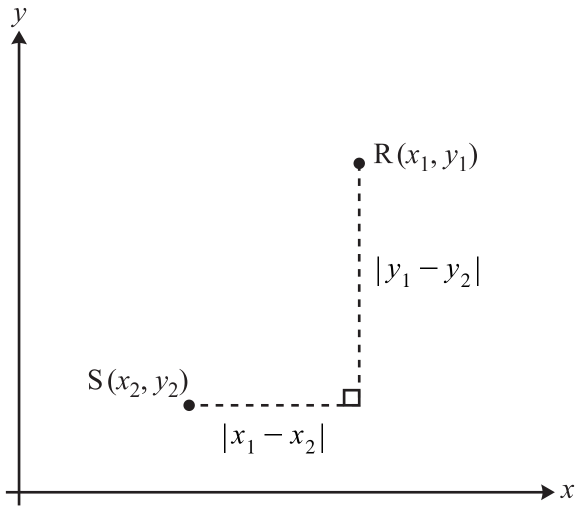

Fig. 6

The taxicab distance, $\mathrm{t}(\mathrm{R},\mathrm{S})$, is defined as $|x_1 - x_2| + |y_1 - y_2|$. Thus the taxicab distance is still defined as the sum of the distances between the points in the x- and y-directions.

For example, the taxicab distance between the points with coordinates $(2.1, 1)$ and $(3.9, 4.3)$ is

$$\begin{aligned} & |2.1 - 3.9| + |1 - 4.3| \\ =\ & |-1.8| + |-3.3| \\ =\ & 1.8 + 3.3 \\ =\ & 5.1. \end{aligned}$$

Similarly the taxicab distance between $(-1.1, 1.4)$ and $(3.2, -0.8)$ is

$$|-1.1 - 3.2| + |1.4 - (-0.8)| = |-4.3| + |2.2| = 4.3 + 2.2 = 6.5.$$

This definition of distance produces some surprising geometric results, as will be seen below.

Fig. 7 shows a fixed point $C(2,3)$ and the locus of the point P satisfying $t(P,C) = 5$. The coordinates of every point $P(x,y)$ on this locus satisfy the equation $|x-2|+|y-3| = 5$.

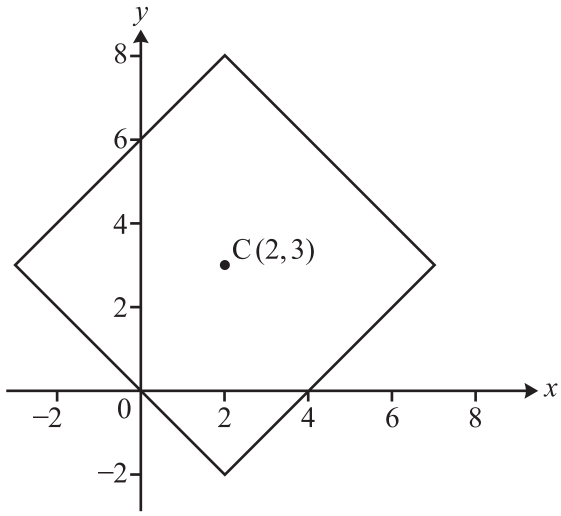

Fig. 7

Since all points are at a fixed taxicab distance from C, this is a taxicab ‘circle’ in this geometry! The circle has a taxicab ‘radius’ of 5.

Furthermore, adding a second taxicab circle with centre $(2,0)$ and radius 2 shows that in generalised taxicab geometry two different circles can have an infinite number of points in common!

Now consider the locus of a point Q which is ‘equidistant’ from two fixed points $A(0,0)$ and $B(8,6)$.

Fig. 8.1 shows the set of points Q satisfying $d(Q,A) = d(Q,B)$; this is the familiar perpendicular bisector of the line segment AB.

Fig. 8.2 shows the set of points Q satisfying $t(Q,A) = t(Q,B)$; so in generalised taxicab geometry the locus is quite different.

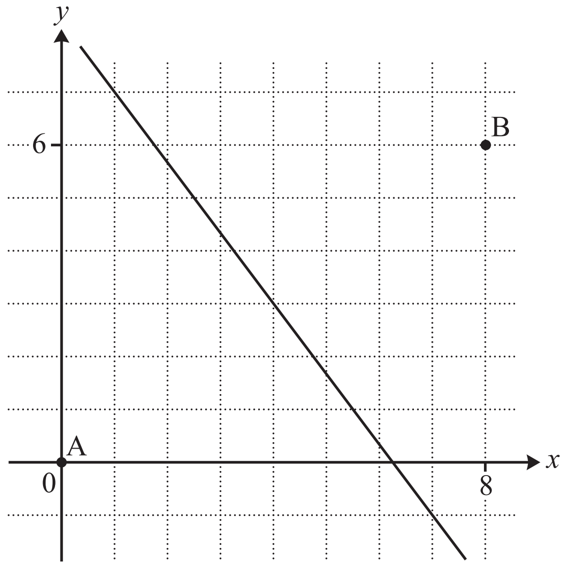

Fig. 8.1

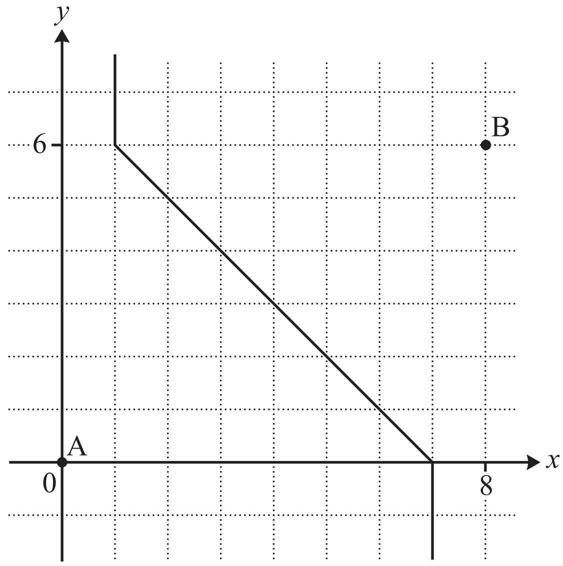

Fig. 8.2

Conclusion

In the natural world it is often appropriate to apply Pythagoras's Theorem to calculate the distance between two points. However, in urban geography, where there are obstacles such as buildings to be considered, taxicab geometry is often a more useful mathematical model.

In this article several simplifying assumptions have been made. For example, the imaginary town is laid out in a square grid, all roads are traversable in both directions and that the rate of progress along every route is uniform. Although these clearly do not exactly match any real cities, Fig. 9 below, a map of Manhattan in New York, suggests that, for some cities, some form of taxicab geometry can provide a good mathematical model.

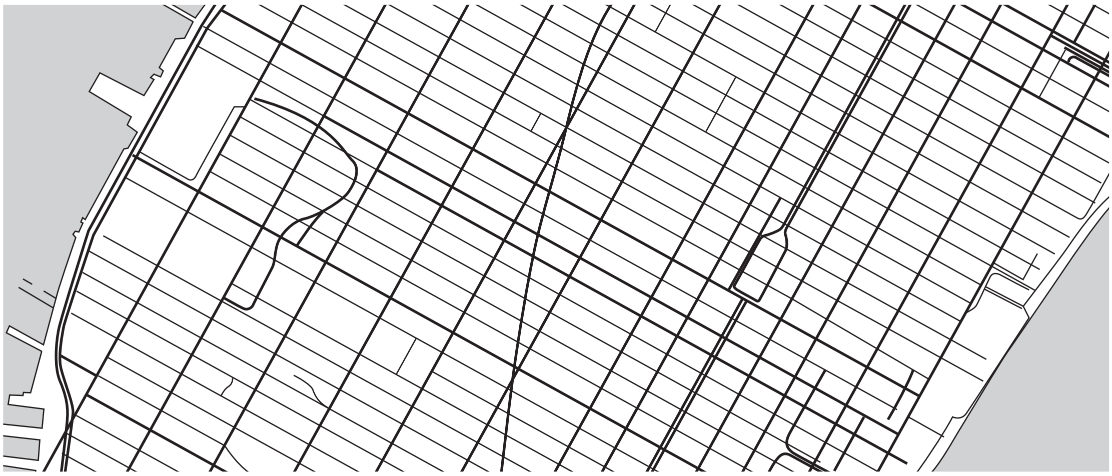

Fig. 9