Oxford Cambridge and RSA

Wednesday 15 June 2022 – Afternoon

A Level Further Mathematics B (MEI)

Y421/01 Mechanics Major

Time allowed: 2 hours 15 minutes

9451680231

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

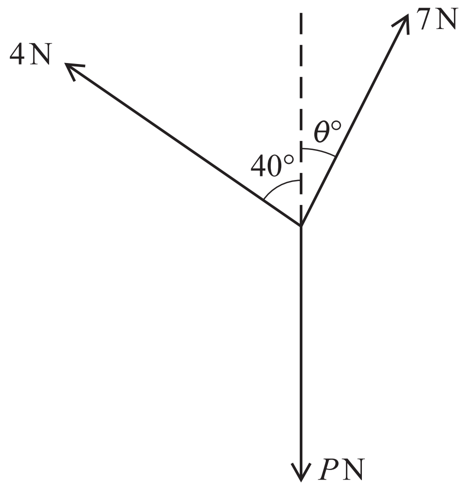
INFORMATION
- The total mark for this paper is **120**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Answer **all** the questions.

Section A (29 marks)

1



Three forces of magnitudes 4 N, 7 N and P N act at a point in the directions shown in the diagram.

The forces are in equilibrium.

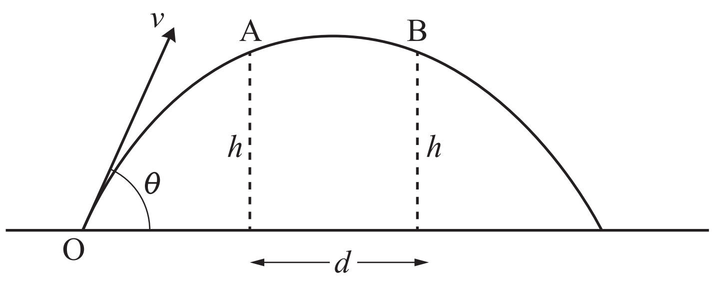
(a) Draw a closed figure to represent the three forces. **[1]**

(b) Hence, or otherwise, find the following.

(i) The value of θ. **[2]**

(ii) The value of P. **[2]**

2



A particle is projected with speed v from a point O on horizontal ground. The angle of projection is θ above the horizontal. The particle passes, in succession, through two points A and B, each at a height h above the ground and a distance d apart, as shown in the diagram.

You are given that $d^2 = \dfrac{v^\alpha \sin^2 2\theta}{g^\beta} - \dfrac{8hv^2\cos^2\theta}{g}$.

Use dimensional analysis to find α and β. **[4]**

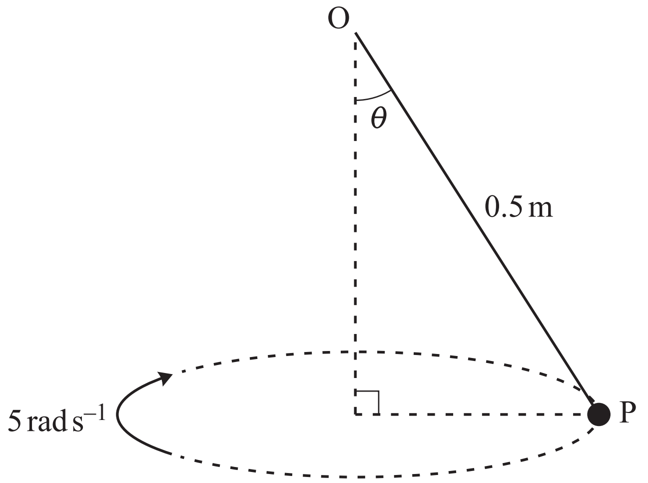
3 A particle, of mass 2 kg, is placed at a point A on a rough horizontal surface. There is a straight vertical wall on the surface and the point on the wall nearest to A is B. The distance AB is 5 m. The particle is projected with speed $4.2\,\text{m}\,\text{s}^{-1}$ along the surface from A towards B. The particle hits the wall directly and rebounds. The coefficient of friction between the particle and the surface is 0.1.

(a) Determine the speed of the particle immediately before impact with the wall. **[4]**

The magnitude of the impulse that the wall exerts on the particle is 9.8 N s.

(b) Find the speed of the particle immediately after impact with the wall. **[2]**

4



The diagram shows a particle P, of mass 0.1 kg, which is attached by a light inextensible string of length 0.5 m to a fixed point O.

P moves with constant angular speed $5\,\text{rad}\,\text{s}^{-1}$ in a horizontal circle with centre vertically below O. The string is inclined at an angle θ to the vertical.

(a) Determine the tension in the string. **[3]**

(b) Find the value of θ. **[2]**

(c) Find the kinetic energy of P. **[2]**

5 At time t seconds, where $t \geqslant 0$, a particle P of mass 2 kg is moving on a smooth horizontal surface. The particle moves under the action of a constant horizontal force of $(-2\mathbf{i} + 6\mathbf{j})$ N and a variable horizontal force of $(2\cos 2t\,\mathbf{i} + 4\sin t\,\mathbf{j})$ N. The acceleration of P at time t seconds is $\mathbf{a}\,\text{m}\,\text{s}^{-2}$.

(a) Find $\mathbf{a}$ in terms of t. **[2]**

The particle P is at rest when $t = 0$.

(b) Determine the speed of P at the instant when $t = 2$. **[5]**

Answer **all** the questions.

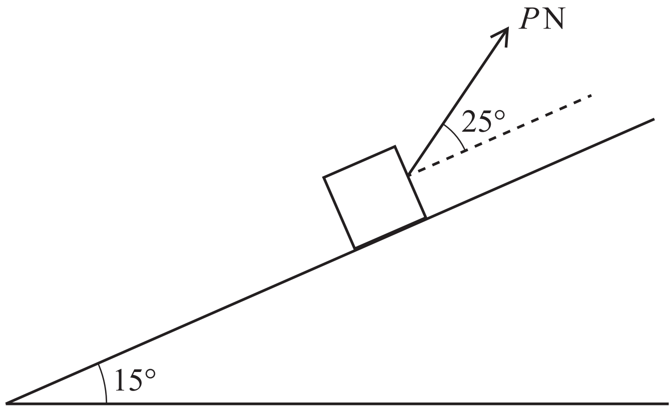
Section B (91 marks)

6 **In this question the box should be modelled as a particle.**

A box of mass m kg is placed on a rough slope which makes an angle of α with the horizontal.

(a) Show that the box is on the point of slipping if $\mu = \tan\alpha$, where μ is the coefficient of friction between the box and the slope. **[2]**

A box of mass 5 kg is pulled up a rough slope which makes an angle of 15° with the horizontal. The box is subject to a constant frictional force of magnitude 3 N. The speed of the box increases from $2\,\mathrm{m\,s^{-1}}$ at a point A on the slope to $5\,\mathrm{m\,s^{-1}}$ at a point B on the slope with B higher up the slope than A. The distance AB is 10 m.



The pulling force has constant magnitude P N and acts at a constant angle of 25° above the slope, as shown in the diagram.

(b) Use the work–energy principle to determine the value of P. **[5]**

7 Two small uniform smooth spheres A and B, of masses 2 kg and 3 kg respectively, are moving in opposite directions along the same straight line towards each other on a smooth horizontal surface. Sphere A has speed $2\,\mathrm{m\,s^{-1}}$ and B has speed $1\,\mathrm{m\,s^{-1}}$ before they collide. The coefficient of restitution between A and B is e.

(a) Show that the velocity of B after the collision, in the original direction of motion of A, is $\frac{1}{5}(1+6e)\,\mathrm{m\,s^{-1}}$ and find a similar expression for the velocity of A after the collision. **[5]**

(b) The following three parts are independent of each other, and each considers a different scenario regarding the collision between A and B.

(i) In the collision between A and B the spheres coalesce to form a combined body C. State the speed of C after the collision. **[1]**

(ii) In the collision between A and B the direction of motion of A is reversed. Find the range of possible values of e. **[2]**

(iii) The total loss in kinetic energy due to the collision is 3 J. Determine the value of e. **[4]**

8 A particle P is projected from a fixed point O with initial velocity $u\mathbf{i}+ku\mathbf{j}$, where k is a positive constant. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are in the horizontal and vertically upward directions respectively. P moves with constant gravitational acceleration of magnitude g.
At time $t \geqslant 0$, particle P has position vector $\mathbf{r}$ relative to O.

(a) Starting from an expression for $\ddot{\mathbf{r}}$, use integration to derive the formula

$$\mathbf{r} = ut\mathbf{i}+\left(kut-\tfrac{1}{2}gt^2\right)\mathbf{j}.$$ **[4]**

The position vector $\mathbf{r}$ of P at time $t \geqslant 0$ can be expressed as $\mathbf{r} = x\mathbf{i}+y\mathbf{j}$, where the axes O$x$ and Oy are horizontally and vertically upwards through O respectively. The axis Ox lies on horizontal ground.

(b) Show that the path of P has cartesian equation

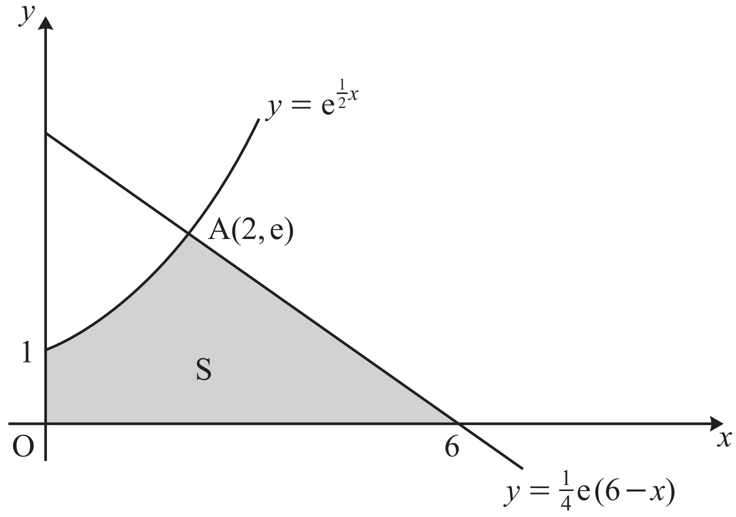
$$gx^2-2ku^2x+2u^2y=0.$$ **[3]**

(c) Hence find, in terms of g, k and u, the maximum height of P above the ground during its motion. **[3]**

The maximum height P reaches above the ground is equal to the distance OA, where A is the point where P first hits the ground.

(d) Determine the value of k. **[3]**

9 [In this question you may use the facts that for a uniform solid right circular cone of height h and base radius r the volume is $\frac{1}{3}\pi r^2 h$ and the centre of mass is $\frac{1}{4}h$ above the base on the line from the centre of the base to the vertex.]



The diagram shows the shaded region S bounded by the curve $y = e^{\frac{1}{2}x}$ for $0 \leqslant x \leqslant 2$, the x-axis, the y-axis, and the line $y = \frac{1}{4}e(6 - x)$.

The line $y = \frac{1}{4}e(6 - x)$ meets the curve $y = e^{\frac{1}{2}x}$ at the point A with coordinates $(2, e)$.

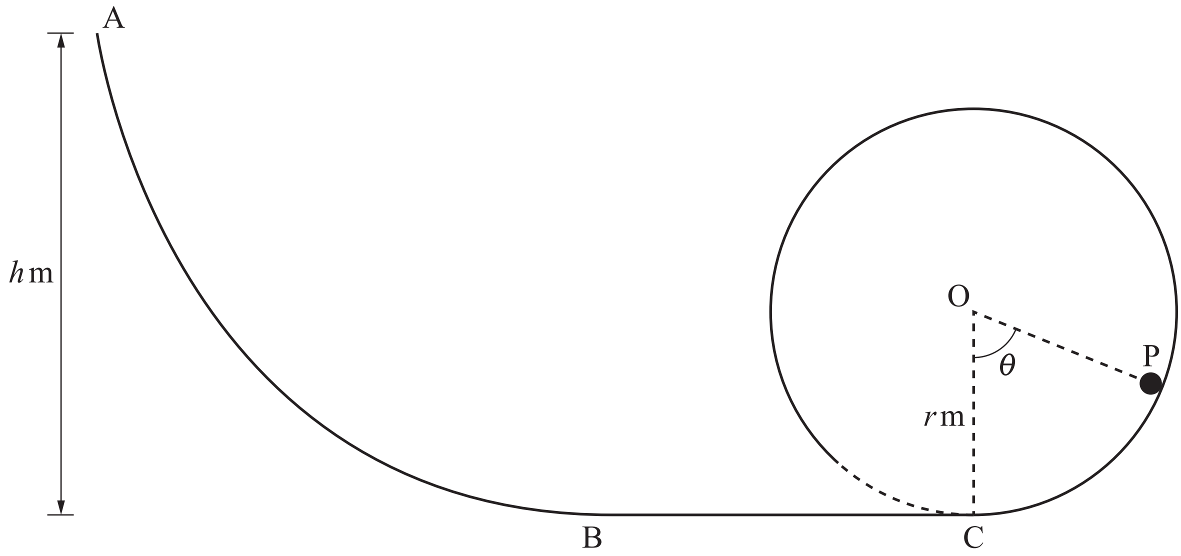
The region S is rotated through 2π radians about the x-axis to form a uniform solid of revolution T.

(a) Show that the x-coordinate of the centre of mass of T is $\dfrac{3(5e^2 + 1)}{7e^2 - 3}$. **[8]**

Solid T is freely suspended from A and hangs in equilibrium.

(b) Determine the angle between AO, where O is the origin, and the vertical. **[3]**

10



A small toy car runs along a track in a vertical plane.

The track consists of three sections: a curved section AB, a horizontal section BC which rests on the floor, and a circular section that starts at C with centre O and radius r m.

The section BC is tangential to the curved section at B and tangential to the circular section at C, as shown in the diagram.

The car, of mass m kg, is placed on the track at A, at a height h m above the floor, and released from rest. The car runs along the track from A to C and enters the circular section at C. It can be assumed that the track does not obstruct the car moving on to the circular section at C.

The track is modelled as being smooth, and the car is modelled as a particle P.

(a) Show that, while P remains in contact with the circular section of the track, the magnitude of the normal contact force between P and the circular section is

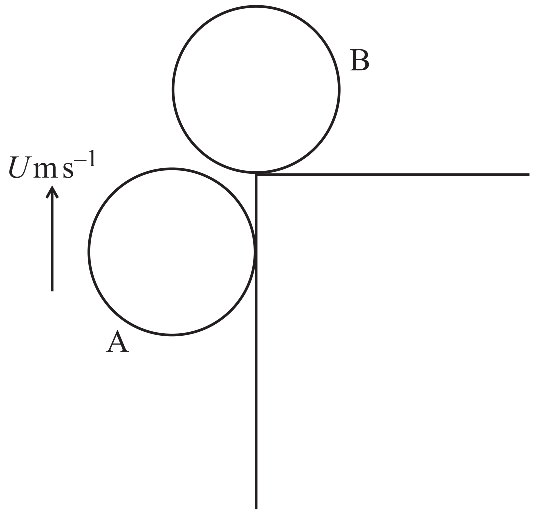
$$mg\left(3\cos\theta - 2 + \frac{2h}{r}\right)\text{N},$$

where θ is the angle between OC and OP. **[7]**

(b) Hence determine, in terms of r, the least possible value of h so that P can complete a vertical circle. **[2]**

(c) Apart from not modelling the car as a particle, state one refinement that would make the model more realistic. **[1]**

11



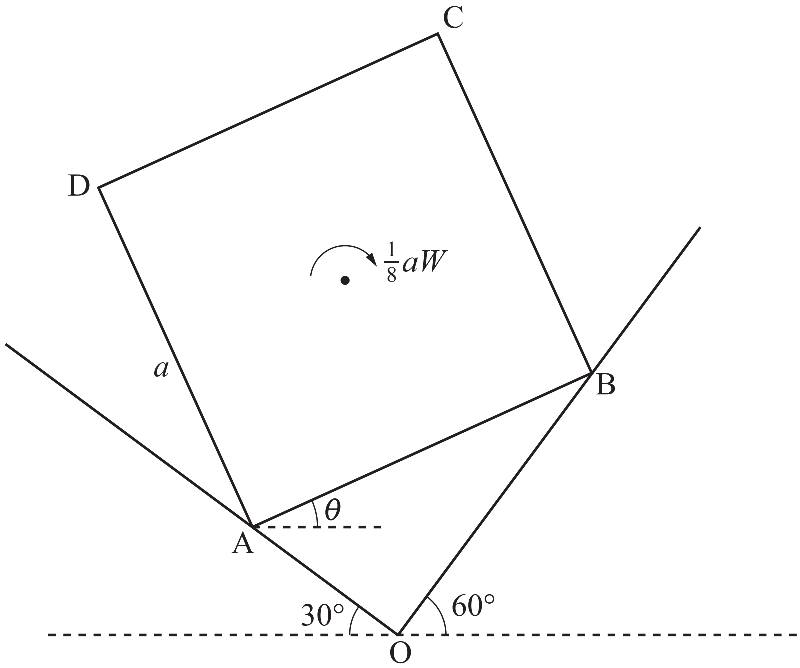
The diagram shows two small identical uniform smooth spheres, A and B, just before A collides with B. Sphere B is at rest on a horizontal table with its centre vertically above the edge of the table. Sphere A is projected vertically upwards so that, just before it collides with B, the speed of A is $U\,\text{m s}^{-1}$ and it is in contact with the vertical side of the table. The point of contact of A with the vertical side of the table and the centres of the spheres are in the same vertical plane.

(a) Show that on impact the line of centres makes an angle of 30° with the vertical. **[1]**

The coefficient of restitution between A and B is $\frac{1}{2}$. After the impact B moves freely under gravity.

(b) Determine, in terms of U and g, the time taken for B to first return to the table. **[7]**

12



The diagram shows a uniform square lamina ABCD, of weight W and side-length a. The lamina is in equilibrium in a vertical plane that also contains the point O. The vertex A rests on a smooth plane inclined at an angle of 30° to the horizontal. The vertex B rests on a smooth plane inclined at an angle of 60° to the horizontal.

OA is a line of greatest slope of the plane inclined at 30° to the horizontal and OB is a line of greatest slope of the plane inclined at 60° to the horizontal.

The side AB is inclined at an angle θ to the horizontal and the lamina is kept in equilibrium in this position by a clockwise couple of magnitude $\frac{1}{8}aW$.

(a) By resolving horizontally and vertically, determine, in terms of W, the magnitude of the normal contact force between the plane and the lamina at B. **[6]**

(b) By taking moments about A, show that θ satisfies the equation

$$2(\sqrt{3}+2)\sin\theta - 2\cos\theta = 1.$$ **[5]**

(c) Verify that $\theta = 22.4°$, correct to **1** decimal place. **[2]**

13 In this question take $g = 10$.

A particle P of mass 0.15 kg is attached to one end of a light elastic string of modulus of elasticity 13.5 N and natural length 0.45 m. The other end of the string is attached to a fixed point O. The particle P rests in equilibrium at a point A with the string vertical.

(a) Show that the distance OA is 0.5 m. **[3]**

At time $t = 0$, P is projected vertically downwards from A with a speed of $1.25\,\text{m}\,\text{s}^{-1}$. Throughout the subsequent motion, P experiences a variable resistance R newtons which is of magnitude 0.6 times its speed (in $\text{m}\,\text{s}^{-1}$).

(b) Given that the downward displacement of P from A at time t seconds is x metres, show that, while the string remains taut, x satisfies the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 200x = 0.$$ **[3]**

(c) Verify that $x = \frac{5}{56}\mathrm{e}^{-2t}\sin(14t)$. **[6]**

(d) Determine whether the string becomes slack during the motion. **[5]**

END OF QUESTION PAPER