Oxford Cambridge and RSA

Wednesday 13 October 2021 – Afternoon

A Level Mathematics A

H240/02 Pure Mathematics and Statistics

Time allowed: 2 hours

You must have:
- the Printed Answer Booklet
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **100**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **12** pages.

ADVICE
- Read each question carefully before you start your answer.

Formulae
A Level Mathematics A (H240)

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

Geometric series

$S_n = \dfrac{a(1-r^n)}{1-r}$

$S_\infty = \dfrac{a}{1-r}$ for $|r| < 1$

Binomial series

$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N}),$

where ${}^nC_r = {}_nC_r = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{R})$

Differentiation

$f(x)$	$f'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

Quotient rule $y = \dfrac{u}{v}, \dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Differentiation from first principles

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Integration

$\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$

$\displaystyle\int f'(x)(f(x))^n\,dx = \frac{1}{n+1}(f(x))^{n+1} + c$

Integration by parts $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

Small angle approximations

$\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, $\tan\theta \approx \theta$ where θ is measured in radians

Trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \mathrm{si}\quad A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad \left(A \pm B \neq (k + \tfrac{1}{2})\pi\right)$

Numerical methods

Trapezium rule: $\int_a^b y\,\mathrm{d}x \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \dfrac{b-a}{n}$

The Newton-Raphson iteration for solving $\mathrm{f}(x) = 0$: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Probability

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A) = \mathrm{P}(B)\mathrm{P}(A \mid B)$ **or** $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

Standard deviation

$\sqrt{\dfrac{\Sigma(x-\overline{x})^2}{n}} = \sqrt{\dfrac{\Sigma x^2}{n} - \overline{x}^2}$ **or** $\sqrt{\dfrac{\Sigma f(x-\overline{x})^2}{\Sigma f}} = \sqrt{\dfrac{\Sigma fx^2}{\Sigma f} - \overline{x}^2}$

The binomial distribution

If $X \sim B(n, p)$ then $P(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$, mean of X is np, variance of X is $np(1-p)$

Hypothesis test for the mean of a normal distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ and $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim \mathrm{N}(0, 1)$

Percentage points of the normal distribution

If Z has a normal distribution with mean 0 and variance 1 then, for each value of p, the table gives the value of z such that $P(Z \leqslant z) = p$.

p	0.75	0.90	0.95	0.975	0.99	0.995	0.9975	0.999	0.9995
z	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

Kinematics

Motion in a straight line

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = \frac{1}{2}(u + v)t$

$v^2 = u^2 + 2as$

$s = vt - \frac{1}{2}at^2$

Motion in two dimensions

$\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = \frac{1}{2}(\mathbf{u} + \mathbf{v})t$

$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$

Section A: Pure Mathematics

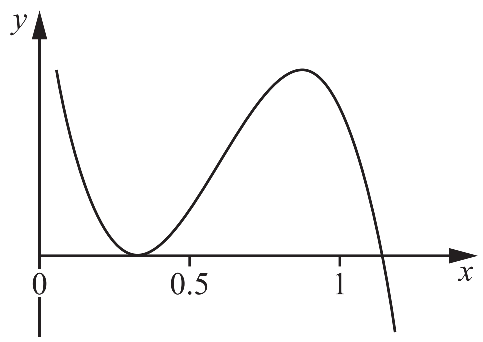
Answer **all** the questions.

1 Differentiate the following with respect to x.

(a) e^{-4x} **[2]**

(b) $\dfrac{x^2}{x+1}$ **[2]**

2 The diagram shows part of the graph of $y = f(x)$, where $f(x)$ is a cubic polynomial in x.



Explain why **one** of the roots of the equation $f(x) = 0$ cannot be found by the sign change method. **[2]**

3 The 15th term of an arithmetic sequence is 88. The sum of the first 10 terms is 310.

Determine the first term and the common difference. **[6]**

4 The size, P, of a population of a certain species of insect at time t months is modelled by the following formula.

$P = 5000 - 1000\cos(30t)°$

(a) Write down the maximum size of the population. **[1]**

(b) Write down the difference between the largest and smallest values of P. **[1]**

(c) Without giving any numerical values, describe briefly the behaviour of the population over time. **[1]**

(d) Find the time taken for the population to return to its initial size for the first time. **[2]**

(e) Determine the time on the second occasion when $P = 4500$. **[4]**

A scientist observes the population over a period of time. He notices that, although the population varies in a way similar to the way predicted by the model, the variations become smaller and smaller over time, and P converges to 5000.

(f) Suggest a change to the model that will take account of this observation. **[1]**

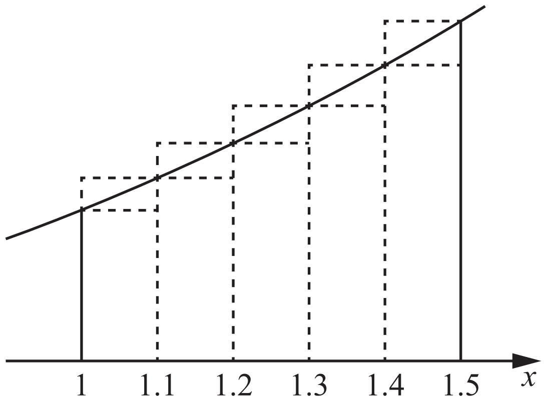
5 **In this question you must show detailed reasoning.**

Points A, B and C have coordinates (0, 6), (7, 5) and (6, −2) respectively.

(a) Find an equation of the perpendicular bisector of AB. **[3]**

(b) Hence, or otherwise, find an equation of the circle that passes through points A, B and C. **[5]**

6 Alex is investigating the area, A, under the graph of $y = x^2$ between $x = 1$ and $x = 1.5$. They draw the graph, together with rectangles of width $\delta x = 0.1$, and varying heights y.



(a) Use the rectangles in the diagram to show that lower and upper bounds for the area A are 0.73 and 0.855 respectively. **[1]**

(b) Alex finds lower and upper bounds for the area A, using widths δx of decreasing size.
The results are shown in the table. Where relevant, values are given correct to 3 significant figures.

Width δx	0.1	0.05	0.025	0.0125
Lower bound for area A	0.73	0.761	0.776	0.784
Upper bound for area A	0.855	0.823	0.807	0.799

Use Alex's results to estimate the value of A correct to **2** significant figures. Give a brief justification for your estimate. **[2]**

(c) Write down an expression, in terms of y and δx, for the exact value of the area A. **[2]**

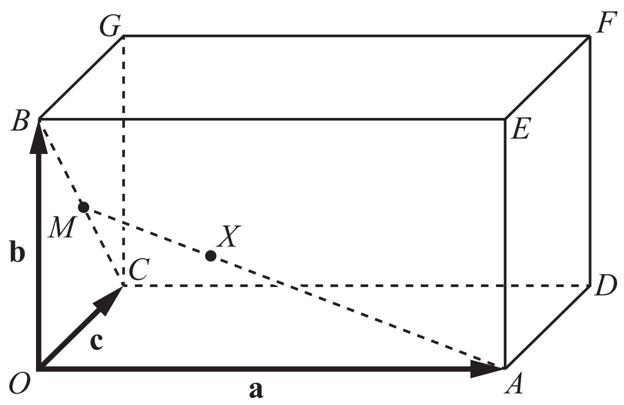
7 Differentiate $\cos x$ with respect to x, from first principles. **[4]**

8 The number K is defined by $K = n^3 + 1$, where n is an integer greater than 2.

(a) Given that $n^3 + 1 \equiv (n+1)(n^2 + bn + c)$, find the constants b and c. **[1]**

(b) Prove that K has at least two **distinct** factors other than 1 and K. **[5]**

9 Points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ relative to an origin O in 3-dimensional space. Rectangles $OADC$ and $BEFG$ are the base and top surface of a cuboid.



- The point M is the midpoint of BC.
- The point X lies on AM such that $AX = 2XM$.

(a) Find $\overrightarrow{OX}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, simplifying your answer. **[4]**

(b) Hence show that the lines OF and AM intersect. **[2]**

Section B: Statistics

Answer **all** the questions.

10 A researcher plans to carry out a statistical investigation to test whether there is linear correlation between the time (T weeks) from conception to birth, and the birth weight (W grams) of new-born babies.

(a) Explain why a 1-tail test is appropriate in this context. **[1]**

The researcher records the values of T and W for a random sample of 11 babies. They calculate Pearson's product-moment correlation coefficient for the sample and find that the value is 0.722.

(b) Use the table below to carry out the test at the 1% significance level. **[5]**

Critical values of Pearson's product-moment correlation coefficient.

	1-tail test	5%	2.5%	1%	0.5%
	2-tail test	10%	5%	2.5%	1%
n	10	0.5494	0.6319	0.7155	0.7646
	11	0.5214	0.6021	0.6851	0.7348
	12	0.4973	0.5760	0.6581	0.7079
	13	0.4762	0.5529	0.6339	0.6835

11 Zac is planning to write a report on the music preferences of the students at his college. There is a large number of students at the college.

(a) State one reason why Zac might wish to obtain information from a sample of students, rather than from all the students. **[1]**

(b) Amaya suggests that Zac should use a sample that is stratified by school year.

Give one advantage of this method as compared with random sampling, in this context. **[1]**

Zac decides to take a random sample of 60 students from his college. He asks each student how many hours per week, on average, they spend listening to music during term. From his results he calculates the following statistics.

Mean	Standard deviation	Median	Lower quartile	Upper quartile
21.0	4.20	20.5	18.0	22.9

(c) Sundip tells Zac that, during term, she spends on average 30 hours per week listening to music.

Discuss briefly whether this value should be considered an outlier. **[3]**

(d) Layla claims that, during term, each student spends on average 20 hours per week listening to music. Zac believes that the true figure is higher than 20 hours. He uses his results to carry out a hypothesis test at the 5% significance level.

Assume that the time spent listening to music is normally distributed with standard deviation 4.20 hours.

Carry out the test. **[7]**

12 Anika and Beth are playing a game which consists of several points.

- The probability that Anika will win any point is 0.7.
- The probability that Beth will win any point is 0.3.
- The outcome of each point is independent of the outcome of every other point.

The first player to win two points wins the game.

(a) Write down the probability that the game consists of more than three points. **[1]**

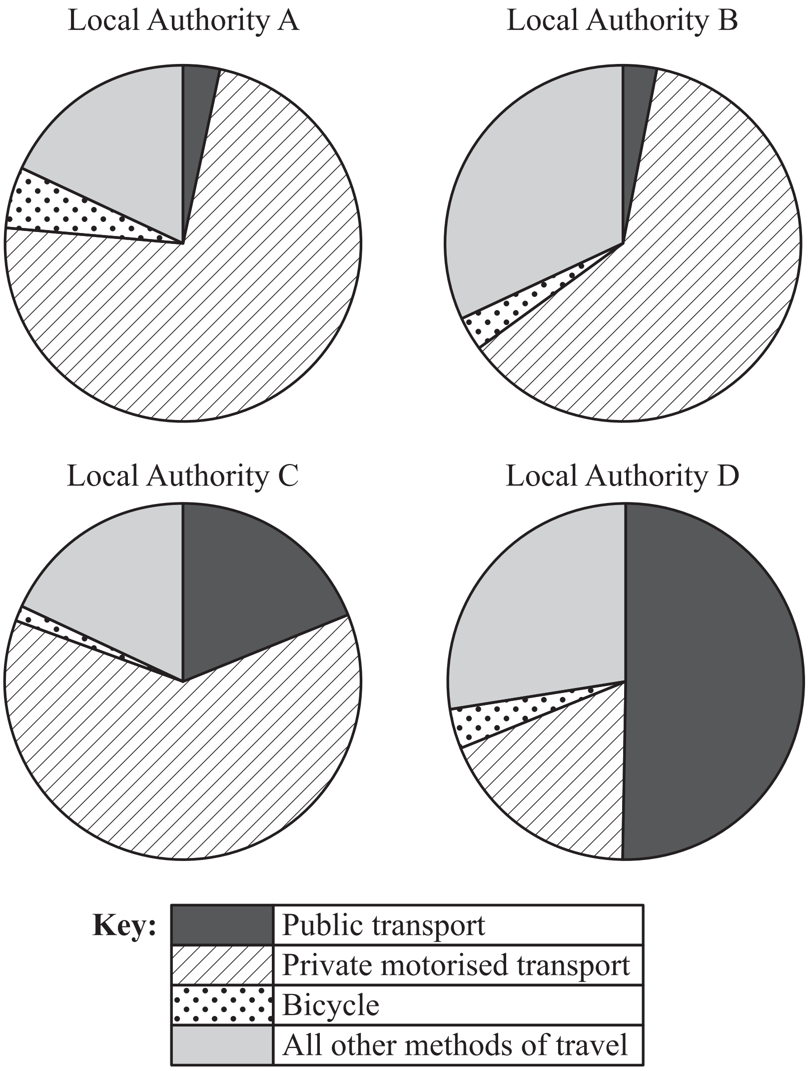
(b) Complete the probability tree diagram in the Printed Answer Booklet showing all the possibilities for the game. **[3]**

(c) Determine the probability that Beth wins the game. **[3]**

(d) Determine the probability that the game consists of exactly three points. **[2]**

(e) Given that Beth wins the game, determine the probability that the game consists of exactly three points. **[4]**

13 The four pie charts illustrate the numbers of employees using different methods of travel in four Local Authorities in 2011.



(a) State, with reasons, which of the four Local Authorities is most likely to be a rural area with many hills. **[2]**

(b) Explain why pie charts are more suitable for answering part **(a)** than bar charts showing the same data. **[1]**

(c) Two of the Local Authorities represent urban areas.

(i) State with a reason which **two** Local Authorities are likely to be urban. **[2]**

(ii) One urban Local Authority introduced a Park-and-Ride service in 2006. Users of this service drive to the edge of the urban area and then use buses to take them into the centre of the area. A student claims that a comparison of the corresponding pie charts for 2001 (not shown) and 2011 would enable them to identify which Local Authority this was.

State with a reason whether you agree with the student. **[2]**

14 The probability distribution of a random variable X is modelled as follows.

$$P(X = x) = \begin{cases} \frac{k}{x} & x = 1, 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Show that $k = \frac{12}{25}$. **[2]**

(b) Show in a table the values of X and their probabilities. **[1]**

(c) The values of three independent observations of X are denoted by X_1, X_2 and X_3.

Find $P(X_1 > X_2 + X_3)$. **[3]**

In a game, a player notes the values of successive independent observations of X and keeps a running total. The aim of the game is to reach a total of exactly 7.

(d) Determine the probability that a total of exactly 7 is first reached on the 5th observation. **[5]**

END OF QUESTION PAPER